MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

State-Space Representation of LTI Systems

1 Introduction

The classical control theory and methods (such as root locus) that we have been using in class to date are based on a simple input-output description of the plant, usually expressed as a transfer function. These methods do not use any knowledge of the interior structure of the plant, and limit us to single-input single-output (SISO) systems, and as we have seen allows only limited control of the closed-loop behavior when feedback control is used.

Modern control theory solves many of the limitations by using a much "richer" description of the plant dynamics. The so-called state-space description provide the dynamics as a set of coupled first-order differential equations in a set of internal variables known as *state variables*, together with a set of algebraic equations that combine the state variables into physical output variables.

1.1 Definition of System State

The concept of the *state* of a dynamic system refers to a minimum set of variables, known as *state variables*, that fully describe the system and its response to any given set of inputs [1-3]. In particular a *state-determined* system model has the characteristic that:

> A mathematical description of the system in terms of a minimum set of variables $x_i(t)$, $i = 1, \ldots, n$, together with knowledge of those variables at an initial time t_0 and the system inputs for time $t \geq t_0$, are sufficient to predict the future system state and outputs for all time $t > t_0$.

This definition asserts that the dynamic behavior of a state-determined system is completely characterized by the response of the set of n variables $x_i(t)$, where the number n is defined to be the *order* of the system.

The system shown in Fig. 1 has two inputs $u_1(t)$ and $u_2(t)$, and four output variables $y_1(t), \ldots, y_4(t)$. If the system is state-determined, knowledge of its state variables $(x_1(t_0), x_2(t_0), \ldots, x_n(t_0))$ at some initial time t_0, and the inputs $u_1(t)$ and $u_2(t)$ for $t \geq t_0$ is sufficient to determine all future behavior of the system. The state variables are an *internal* description of the system which completely characterize the system state at any time t, and from which any output variables $y_i(t)$ may be computed.

Large classes of engineering, biological, social and economic systems may be represented by state-determined system models. System models constructed with the pure and ideal (linear) one-port elements (such as mass, spring and damper elements) are state-determined

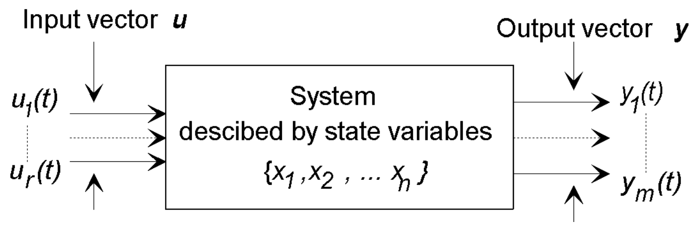

Figure 1: System inputs and outputs.

system models. For such systems the number of state variables, n, is equal to the number of *independent* energy storage elements in the system. The values of the state variables at any time t specify the energy of each energy storage element within the system and therefore the total system energy, and the time derivatives of the state variables determine the rate of change of the system energy. Furthermore, the values of the system state variables at any time t provide sufficient information to determine the values of all other variables in the system at that time.

There is no unique set of state variables that describe any given system; many different sets of variables may be selected to yield a complete system description. However, for a given system the order n is unique, and is independent of the particular set of state variables chosen. State variable descriptions of systems may be formulated in terms of physical and measurable variables, or in terms of variables that are not directly measurable. It is possible to mathematically transform one set of state variables to another; the important point is that any set of state variables must provide a complete description of the system. In this note we concentrate on a particular set of state variables that are based on energy storage variables in physical systems.

1.2 The State Equations

A standard form for the state equations is used throughout system dynamics. In the standard form the mathematical description of the system is expressed as a set of n coupled first-order ordinary differential equations, known as the *state equations*, in which the time derivative of each state variable is expressed in terms of the state variables $x_1(t), \ldots, x_n(t)$ and the system inputs $u_1(t), \ldots, u_r(t)$. In the general case the form of the n state equations is:

$$\begin{aligned} \dot{x_1} &= f_1(\mathbf{x}, \mathbf{u}, t) \\ \dot{x_2} &= f_2(\mathbf{x}, \mathbf{u}, t) \\ \vdots &= \vdots \\ \dot{x_n} &= f_n(\mathbf{x}, \mathbf{u}, t) \end{aligned} \tag{1}$$

where $\dot{x_i} = dx_i/dt$ and each of the functions $f_i(\mathbf{x}, \mathbf{u}, t)$, $(i = 1, \ldots, n)$ may be a general nonlinear, time varying function of the state variables, the system inputs, and time. [1]

It is common to express the state equations in a vector form, in which the set of n state variables is written as a *state vector* $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$, and the set of r inputs is written as an input vector $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^T$. Each state variable is a time varying component of the column vector $\mathbf{x}(t)$.

This form of the state equations explicitly represents the basic elements contained in the definition of a state determined system. Given a set of initial conditions (the values of the x_i at some time t_0) and the inputs for $t \geq t_0$, the state equations explicitly specify the derivatives of all state variables. The value of each state variable at some time Δt later may then be found by direct integration.

The system state at any instant may be interpreted as a point in an n-dimensional *state space*, and the dynamic state response $\mathbf{x}(t)$ can be interpreted as a path or trajectory traced out in the state space.

In vector notation the set of n equations in Eqs. (1) may be written:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t). \tag{2}$$

where $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is a *vector* function with n components $f_i(\mathbf{x}, \mathbf{u}, t)$.

In this note we restrict attention primarily to a description of systems that are *linear* and *time-invariant* (LTI), that is systems described by linear differential equations with constant coefficients. For an LTI system of order n, and with r inputs, Eqs. (1) become a set of n coupled first-order linear differential equations with constant coefficients:

$$\begin{array}{lcccccccccccccc}
\dot{x_1} & = & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & + & b_{11}u_1 & + & \ldots & + & b_{1r}u_r \\
\dot{x_2} & = & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & + & b_{21}u_1 & + & \ldots & + & b_{2r}u_r \\
\vdots & & \vdots & & & & & & & & & & & & \\
\dot{x_n} & = & a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & + & b_{n1}u_1 & + & \ldots & + & b_{nr}u_r
\end{array} \tag{3}$$

where the coefficients a_{ij} and b_{ij} are constants that describe the system. This set of n equations defines the derivatives of the state variables to be a weighted sum of the state variables and the system inputs.

Equations (8) may be written compactly in a matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \ldots & b_{1r} \\ b_{21} & & b_{2r} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix} \tag{4}$$

which may be summarized as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5}$$

[1]In this note we use bold-faced type to denote vector quantities. Upper case letters are used to denote general matrices while lower case letters denote column vectors. See Appendix A for an introduction to matrix notation and operations.

where the state vector $\mathbf{x}$ is a column vector of length n, the input vector $\mathbf{u}$ is a column vector of length r, $\mathbf{A}$ is an $n \times n$ square matrix of the constant coefficients a_{ij}, and $\mathbf{B}$ is an $n \times r$ matrix of the coefficients b_{ij} that weight the inputs.

1.3 Output Equations

A system *output* is defined to be any system variable of interest. A description of a physical system in terms of a set of state variables does not necessarily include all of the variables of direct engineering interest. An important property of the linear state equation description is that all system variables may be represented by a linear combination of the state variables x_i and the system inputs u_i. An arbitrary output variable in a system of order n with r inputs may be written:

$$y(t) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + d_1 u_1 + \ldots + d_r u_r \tag{6}$$

where the c_i and d_i are constants. If a total of m system variables are defined as outputs, the m such equations may be written as:

$$\begin{array}{ccccccccccccccc} y_1 & = & c_{11}x_1 & + & c_{12}x_2 & + & \ldots & + & c_{1n}x_n & + & d_{11}u_1 & + & \ldots & + & d_{1r}u_r \\ y_2 & = & c_{21}x_1 & + & c_{22}x_2 & + & \ldots & + & c_{2n}x_n & + & d_{21}u_1 & + & \ldots & + & d_{2r}u_r \\ \vdots & & \vdots & & & & & & & & & & & & \\ y_m & = & c_{m1}x_1 & + & c_{m2}x_2 & + & \ldots & + & c_{mn}x_n & + & d_{m1}u_1 & + & \ldots & + & d_{mr}u_r \end{array} \tag{7}$$

or in matrix form:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & & & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & \ldots & d_{1r} \\ d_{21} & & d_{2r} \\ \vdots & & \vdots \\ d_{m1} & \ldots & d_{mr} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}. \tag{8}$$

The output equations, Eqs. (8), are commonly written in the compact form:

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{9}$$

where $\mathbf{y}$ is a column vector of the output variables $y_i(t)$, $\mathbf{C}$ is an $m \times n$ matrix of the constant coefficients c_{ij} that weight the state variables, and D is an $m \times r$ matrix of the constant coefficients d_{ij} that weight the system inputs. For many physical systems the matrix $\mathbf{D}$ is the null matrix, and the output equation reduces to a simple weighted combination of the state variables:

$$\mathbf{y} = \mathbf{C}\mathbf{x}. \tag{10}$$

1.4 State Equation Based Modeling Procedure

The complete system model for a linear time-invariant system consists of (i) a set of n state equations, defined in terms of the matrices **A** and **B**, and (ii) a set of output equations that relate any output variables of interest to the state variables and inputs, and expressed in terms of the **C** and **D** matrices. The task of modeling the system is to derive the elements of the matrices, and to write the system model in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (11)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}. \quad (12)$$

The matrices **A** and **B** are properties of the system and are determined by the system structure and elements. The output equation matrices **C** and **D** are determined by the particular choice of output variables.

The overall modeling procedure developed in this chapter is based on the following steps:

1. Determination of the system order n and selection of a set of state variables from the linear graph system representation.

2. Generation of a set of state equations and the system **A** and **B** matrices using a well defined methodology. This step is also based on the linear graph system description.

3. Determination of a suitable set of output equations and derivation of the appropriate **C** and **D** matrices.

2 Block Diagram Representation of Linear Systems Described by State Equations

The matrix-based state equations express the derivatives of the state-variables explicitly in terms of the states themselves and the inputs. In this form, the state vector is expressed as the direct result of a vector integration. The block diagram representation is shown in Fig. 2. This general block diagram shows the matrix operations from input to output in terms of the **A**, **B**, **C**, **D** matrices, but does not show the path of individual variables.

In state-determined systems, the state variables may always be taken as the outputs of integrator blocks. A system of order n has n integrators in its block diagram. The derivatives of the state variables are the inputs to the integrator blocks, and each state equation expresses a derivative as a sum of weighted state variables and inputs. A detailed block diagram representing a system of order n may be constructed directly from the state and output equations as follows:

Step 1: Draw n integrator (S^{-1}) blocks, and assign a state variable to the output of each block.

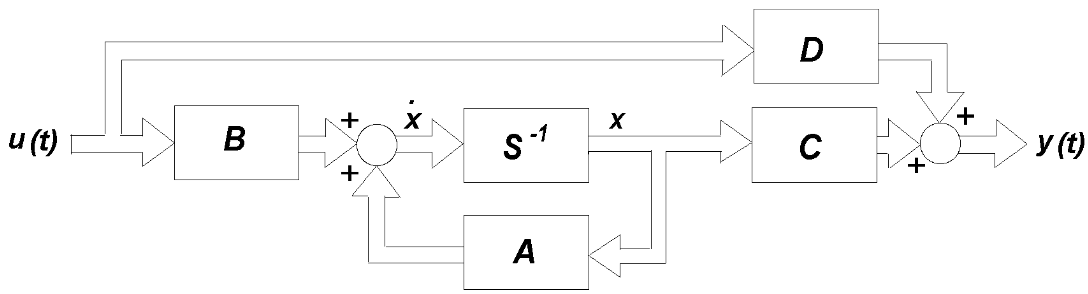

Figure 2: Vector block diagram for a linear system described by state-space system dynamics.

Step 2: At the input to each block (which represents the derivative of its state variable) draw a summing element.

Step 3: Use the state equations to connect the state variables and inputs to the summing elements through scaling operator blocks.

Step 4: Expand the output equations and sum the state variables and inputs through a set of scaling operators to form the components of the output.

■ Example 1

Draw a block diagram for the general second-order, single-input single-output system:

$$\begin{aligned} \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + du(t). \end{aligned} \tag{i}$$

Solution: The block diagram shown in Fig. 3 was drawn using the four steps described above.

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3 Transformation From State-Space Equations to Classical Form

The transfer function and the classical input-output differential equation for any system variable may be found directly from a state space representation through the Laplace transform. The following example illustrates the general method for a first-order system.

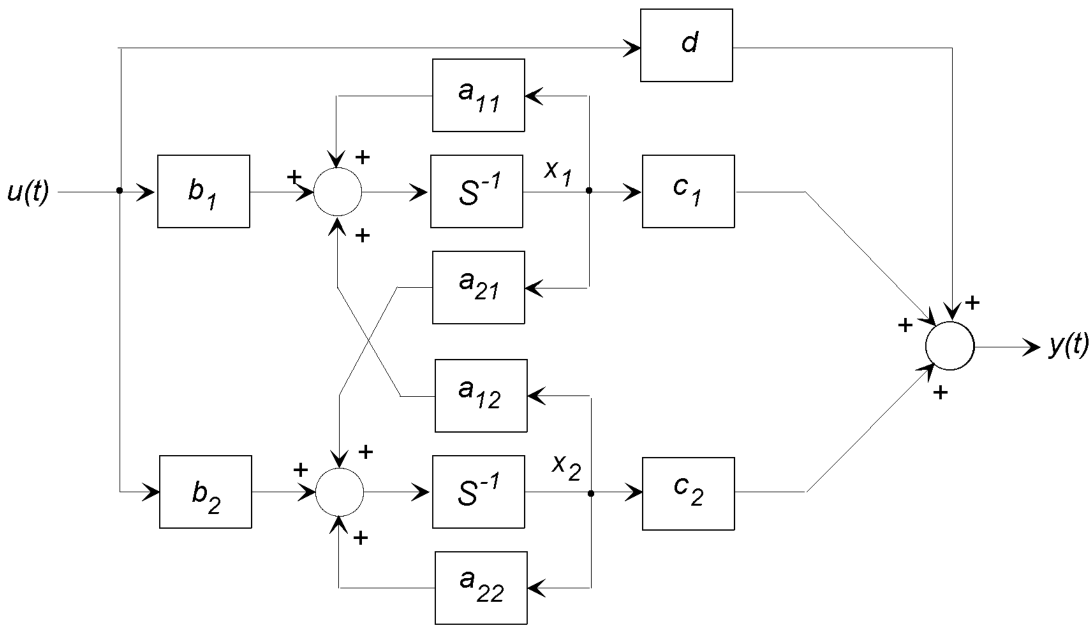

Figure 3: Block diagram for a state-equation based second-order system.

■ Example 2

Find the transfer function and a single first-order differential equation relating the output $y(t)$ to the input $u(t)$ for a system described by the first-order linear state and output equations:

$$\frac{dx}{dt} = ax(t) + bu(t) \quad \text{(i)}$$

$$y(t) = cx(t) + du(t) \quad \text{(ii)}$$

Solution: The Laplace transform of the state equation is

$$sX(s) = aX(s) + bU(s), \quad \text{(iii)}$$

which may be rewritten with the state variable $X(s)$ on the left-hand side:

$$(s - a)\,X(s)) = bU(s). \quad \text{(iv)}$$

Then dividing by $(s - a)$, solve for the state variable:

$$X(s) = \frac{b}{s-a}U(s), \quad \text{(v)}$$

and substitute into the Laplace transform of the output equation $Y(s) = cX(s) + dU(s)$:

$$\begin{aligned} Y(s) &= \left[\frac{bc}{s-a} + d\right] U(s) \\ &= \frac{ds + (bc - ad)}{(s-a)} U(s) \qquad \text{(vi)} \end{aligned}$$

The transfer function is:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{ds + (bc - ad)}{(s-a)}. \qquad \text{(vii)}$$

The differential equation is found directly:

$$(s-a) Y(s) = (ds + (bc - ad)) U(s), \qquad \text{(viii)}$$

and rewriting as a differential equation:

$$\frac{dy}{dt} - ay = d\frac{du}{dt} + (bc - ad) u(t). \qquad \text{(ix)}$$

Classical representations of higher-order systems may be derived in an analogous set of steps by using the Laplace transform and matrix algebra. A set of linear state and output equations written in standard form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (13)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \qquad (14)$$

may be rewritten in the Laplace domain. The system equations are then

$$\begin{aligned} s\mathbf{X}(s) &= \mathbf{A}X(s) + \mathbf{B}U(s) \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \qquad (15) \end{aligned}$$

and the state equations may be rewritten:

$$s\mathbf{x}(s) - \mathbf{A}\mathbf{x}(s) = [s\mathbf{I} - \mathbf{A}]\,\mathbf{x}(s) = \mathbf{B}\mathbf{u}(s). \qquad (16)$$

where the term $s\mathbf{I}$ creates an $n \times n$ matrix with s on the leading diagonal and zeros elsewhere. (This step is necessary because matrix addition and subtraction is only defined for matrices of the same dimension.) The matrix $[s\mathbf{I} - \mathbf{A}]$ appears frequently throughout linear system theory; it is a square $n \times n$ matrix with elements directly related to the $\mathbf{A}$ matrix:

$$[s\mathbf{I} - A] = \begin{bmatrix} (s - a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (s - a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (s - a_{nn}) \end{bmatrix}. \qquad (17)$$

The state equations, written in the form of Eq. (16), are a set of n simultaneous operational expressions. The common methods of solving linear algebraic equations, for example Gaussian elimination, Cramer's rule, the matrix inverse, elimination and substitution, may be directly applied to linear operational equations such as Eq. (16).

For low-order single-input single-output systems the transformation to a classical formulation may be performed in the following steps:

1. Take the Laplace transform of the state equations.
2. Reorganize each state equation so that all terms in the state variables are on the left-hand side.
3. Treat the state equations as a set of simultaneous algebraic equations and solve for those state variables required to generate the output variable.
4. Substitute for the state variables in the output equation.
5. Write the output equation in operational form and identify the transfer function.
6. Use the transfer function to write a single differential equation between the output variable and the system input.

This method is illustrated in the following two examples.

■ Example 3

Use the Laplace transform method to derive a single differential equation for the capacitor voltage v_C in the series R-L-C electric circuit shown in Fig. 4

Solution: The linear graph method of state equation generation selects the

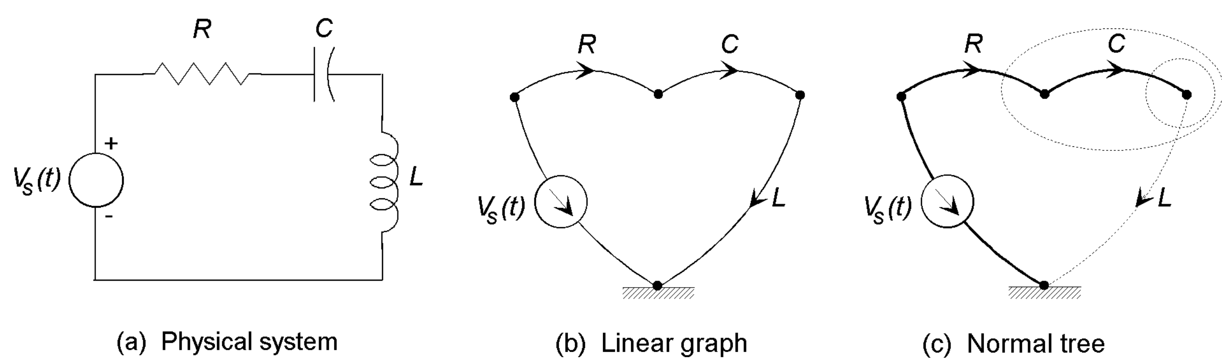

Figure 4: A series RLC circuit.

capacitor voltage $v_C(t)$ and the inductor current $i_L(t)$ as state variables, and generates the following pair of state equations:

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}. \qquad \text{(i)}$$

The required output equation is:

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} V_{in} \qquad \text{(ii)}$$

Step 1: In Laplace transform form the state equations are:

$$\begin{aligned} sV_C(s) &= 0V_C(s) + 1/CI_L(s) + 0V_s(s) \\ sI_L(s) &= -1/LV_C(s) - R/LI_L(s) + 1/LV_s(s) \end{aligned} \qquad \text{(iii)}$$

Step 2: Reorganize the state equations:

$$sV_C(s) - 1/CI_L(s) = 0V_s(s) \qquad \text{(iv)}$$

$$1/LV_C(s) + [s + R/L]\, I_L(s) = 1/LV_s(s) \qquad \text{(v)}$$

Step 3: In this case we have two simultaneous operational equations in the state variables v_C and i_L. The output equation requires only v_C. If Eq. (iv) is multiplied by $[s + R/L]$, and Eq. (v) is multiplied by $1/C$, and the equations added, $I_L(s)$ is eliminated:

$$[s\,(s + R/L) + 1/LC]\, V_C(s) = 1/LCV_s(s) \qquad \text{(vi)}$$

Step 4: The output equation is $y = v_C$. Operate on both sides of Eq. (vi) by $[s^2 + (R/L)s + 1/LC]^{-1}$ and write in quotient form:

$$V_C(s) = \frac{1/LC}{s^2 + (R/L)s + 1/LC} V_s(s) \qquad \text{(vii)}$$

Step 5: The transfer function $H(s) = V_c(s)/V_s(s)$ is:

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + 1/LC} \qquad \text{(viii)}$$

Step 6: The differential equation relating v_C to V_s is:

$$\frac{d^2 v_C}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC} v_C = \frac{1}{LC} V_s(t) \qquad \text{(ix)}$$

Cramer's Rule, for the solution of a set of linear algebraic equations, is a useful method to apply to the solution of these equations. In solving for the variable x_i in a set of n linear algebraic equations, such as $\mathbf{Ax} = \mathbf{b}$ the rule states:

$$x_i = \frac{\det\left[\mathbf{A}^{(i)}\right]}{\det\left[\mathbf{A}\right]} \tag{18}$$

where $\mathbf{A}^{(i)}$ is another $n \times n$ matrix formed by replacing the ith column of $\mathbf{A}$ with the vector $\mathbf{b}$.

If

$$\left[s\mathbf{I} - \mathbf{A}\right]\mathbf{X}(s) = \mathbf{BU}(s) \tag{19}$$

then the relationship between the ith state variable and the input is

$$X_i(s) = \frac{\det\left[\left[s\mathbf{I} - \mathbf{A}\right]^{(i)}\right]}{\det\left[s\mathbf{I} - \mathbf{A}\right]} U(s) \tag{20}$$

where $(s\mathbf{I} - \mathbf{A})^{(i)}$ is defined to be the matrix formed by replacing the ith column of $(s\mathbf{I} - \mathbf{A})$ with the column vector $\mathbf{B}$. The differential equation is

$$\det\left[s\mathbf{I} - \mathbf{A}\right] x_i = \det\left[(s\mathbf{I} - \mathbf{A})^{(i)}\right] u_k(t). \tag{21}$$

■ Example 4

Use Cramer's Rule to solve for $v_L(t)$ in the electrical system of Example 3.

Solution: From Example 3 the state equations are:

$$\begin{bmatrix} \dot{v}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} v_c \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}(t) \tag{i}$$

and the output equation is:

$$v_L = -v_C - Ri_L + V_s(t). \tag{ii}$$

In the Laplace domain the state equations are:

$$\begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix} \begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}(s). \tag{iii}$$

The voltage $V_C(s)$ is given by:

$$\begin{aligned} V_C(s) &= \frac{\det\left[(s\mathbf{I}-\mathbf{A})^{(1)}\right]}{\det\left[(s\mathbf{I}-\mathbf{A})\right]}V_{in}(s) = \frac{\det\begin{bmatrix} 0 & -1/C \\ 1/L & (s+R/L) \end{bmatrix}}{\det\begin{bmatrix} s & -1/C \\ 1/L & (s+R/L) \end{bmatrix}}V_{in}(s) \\ &= \frac{1/LC}{s^2+(R/L)s+(1/LC)}V_{in}(s). \qquad \text{(iv)} \end{aligned}$$

The current $I_L(t)$ is:

$$\begin{aligned} I_L(s) &= \frac{\det\left[(s\mathbf{I}-\mathbf{A})^{(2)}\right]}{\det\left[(s\mathbf{I}-\mathbf{A})\right]}V_{in}(s) = \frac{\det\begin{bmatrix} s & 0 \\ 1/L & 1/L \end{bmatrix}}{\det\begin{bmatrix} s & -1/C \\ 1/L & (s+R/L) \end{bmatrix}}V_{in}(s) \\ &= \frac{s/L}{s^2+(R/L)s+(1/LC)}V_{in}(s). \qquad \text{(v)} \end{aligned}$$

The output equation may be written directly from the Laplace transform of Eq. (ii):

$$\begin{aligned} V_L(s) &= -V_C(s) - RI_L(s) + V_s(s) \\ &= \left[\frac{-1/LC}{s^2+(R/L)s+(1/LC)} + \frac{-(R/L)s}{s^2+(R/L)s+(1/LC)} + 1\right]V_s(s) \\ &= \frac{-1/LC-(R/L)s+(s^2+(R/L)s+(1/LC))}{s^2+(R/L)s+(1/LC)}V_s(s) \\ &= \frac{s^2}{s^2+(R/L)s+(1/LC)}V_s(s), \qquad \text{(vi)} \end{aligned}$$

giving the differential equation

$$\frac{d^2v_L}{dt^2} + \frac{R}{L}\frac{dv_L}{dt} + \frac{1}{LC}v_L(t) = \frac{d^2V_s}{dt^2}. \qquad \text{(vii)}$$

For a single-input single-output (SISO) system the transfer function may be found directly by evaluating the inverse matrix

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}U(s). \qquad (22)$$

Using the definition of the matrix inverse:

$$[s\mathbf{I}-\mathbf{A}]^{-1} = \frac{\operatorname{adj}[s\mathbf{I}-\mathbf{A}]}{\det[s\mathbf{I}-\mathbf{A}]}, \qquad (23)$$

$$\mathbf{X}(s) = \frac{\text{adj}\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{B}}{\det\,[s\mathbf{I} - \mathbf{A}]} U(s). \tag{24}$$

and substituting into the output equations gives:

$$\begin{aligned} Y(s) &= \mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}U(s) + \mathbf{D}U(s) \\ &= \left[\mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}\right] U(s). \end{aligned} \tag{25}$$

Expanding the inverse in terms of the determinant and the adjoint matrix yields:

$$\begin{aligned} Y(S) &= \frac{\mathbf{C}\,\text{adj}\,(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \det\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{D}}{\det\,[s\mathbf{I} - \mathbf{A}]} U(s) \\ &= H(s)U(s) \end{aligned} \tag{26}$$

so that the required differential equation may be found by expanding:

$$\det\,[s\mathbf{I} - \mathbf{A}]\,Y(s) = [\mathbf{C}\,\text{adj}\,(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \det\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{D}]\,U(s). \tag{27}$$

and taking the inverse Laplace transform of both sides.

■ Example 5

Use the matrix inverse method to find a differential equation relating $v_L(t)$ to $V_s(t)$ in the system described in Example 3.

Solution: The state vector, written in the Laplace domain,

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) \tag{i}$$

from the previous example is:

$$\begin{bmatrix} V_c(s) \\ I_L(s) \end{bmatrix} = \begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}(s). \tag{ii}$$

The determinant of $[s\mathbf{I} - \mathbf{A}]$ is

$$\det\,[s\mathbf{I} - \mathbf{A}] = \left(s^2 + (R/L)s + (1/LC)\right), \tag{iii}$$

and the adjoint of $[s\mathbf{I} - \mathbf{A}]$ is

$$\text{adj} \begin{bmatrix} s & -1/C \\ 1/L & s + R/L \end{bmatrix} = \begin{bmatrix} s + R/L & 1/C \\ -1/L & s \end{bmatrix}. \tag{iv}$$

From Example .5 and the previous example, the output equation $v_L(t) = -v_C - Ri_L + V_s(t)$ specifies that $\mathbf{C} = [-1 \;\; -R]$ and $\mathbf{D} = [1]$. The transfer function, Eq. (26) is:

$$H(s) = \frac{\mathbf{C}\,\text{adj}\,(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \det\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{D}}{\det\,[s\mathbf{I} - \mathbf{A}]^{-1}}. \tag{v}$$

Since

$$\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I}-\mathbf{A})\,\mathbf{B} = \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} s+R/L & 1/C \\ -1/L & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$$
$$= -\frac{R}{L}s - \frac{1}{LC}, \qquad \text{(vi)}$$

the transfer function is

$$H(s) = \frac{-(R/L)s - 1/(LC) + (s^2 + (R/L)s + (1/LC)) \begin{bmatrix} 1 \end{bmatrix}}{(s^2 + (R/L)s + (1/LC))}$$
$$= \frac{s^2}{(S^2 + (R/L)S + (1/LC))}, \qquad \text{(vii)}$$

which is the same result found by using Cramer's rule in Example 4.

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4 Transformation from Classical Form to State-Space Representation

The block diagram provides a convenient method for deriving a set of state equations for a system that is specified in terms of a single input/output differential equation. A set of n state variables can be identified as the outputs of integrators in the diagram, and state equations can be written from the conditions at the inputs to the integrator blocks (the derivative of the state variables). There are many methods for doing this; we present here one convenient state equation formulation that is widely used in control system theory.

Let the differential equation representing the system be of order n, and without loss of generality assume that the order of the polynomial operators on both sides is the same:

$$\left(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0\right) Y(s) = \left(b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0\right) U(s). \qquad (28)$$

We may multiply both sides of the equation by s^{-n} to ensure that all differential operators have been eliminated:

$$\left(a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}\right) Y(s) =$$
$$\left(b_n + b_{n-1}s^{-1} + \cdots + b_1 s^{-(n-1)} + \cdots + b_0 s^{-n}\right) U(s), \qquad (29)$$

from which the output may be specified in terms of a transfer function. If we define a dummy variable $Z(s)$, and split Eq. (29) into two parts

$$Z(s) = \frac{1}{a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}} U(s) \qquad (30)$$
$$Y(s) = \left(b_n + b_{n-1}s^{-1} + \cdots + b_1 s^{-(n-1)} + b_0 s^{-n}\right) Z(s), \qquad (31)$$

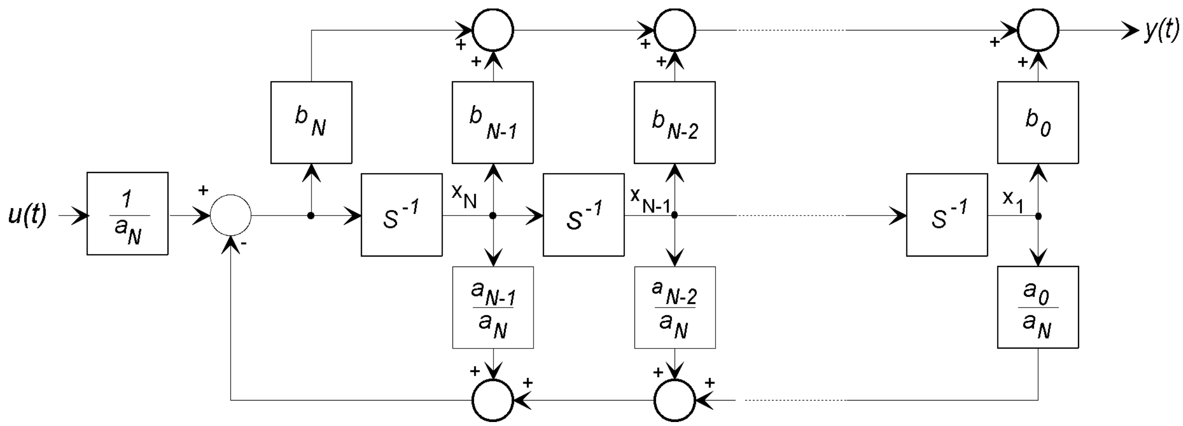

Figure 5: Block diagram of a system represented by a classical differential equation.

Eq. (30) may be solved for $U(s)$),

$$U(s) = \left(a_n + a_{n-1}s^{-1} + \cdots + a_1 s^{-(n-1)} + a_0 s^{-n}\right) X(s) \tag{32}$$

and rearranged to generate a *feedback structure* that can be used as the basis for a block diagram:

$$Z(s) = \frac{1}{a_n}U(s) - \left(\frac{a_{n-1}}{a_n}\frac{1}{s} + \cdots + \frac{a_1}{a_n}\frac{1}{s^{n-1}} + \frac{a_0}{a_n}\frac{1}{s^{-n}}\right) Z(s). \tag{33}$$

The dummy variable $Z(s)$ is specified in terms of the system input $u(t)$ and a weighted sum of successive integrations of itself. Figure 5 shows the overall structure of this direct-form block diagram. A string of n cascaded integrator ($1/s$) blocks, with $Z(s)$ defined at the input to the first block, is used to generate the feedback terms, $Z(s)/s^{-i}, i = 1, \ldots n$, in Eq. (33). Equation (31) serves to combine the outputs from the integrators into the output $y(t)$.

A set of state equations may be found from the block diagram by assigning the state variables $x_i(t)$ to the outputs of the n integrators. Because of the direct cascade connection of the integrators, the state equations take a very simple form. By inspection:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots \quad & \quad \vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + \frac{1}{a_n}u(t).
\end{aligned} \tag{34}$$

In matrix form these equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0/a_n & -a_1/a_n & \cdots & -a_{n-2}/a_n & -a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1/a_n \end{bmatrix} u(t). \tag{35}$$

The **A** matrix has a very distinctive form. Each row, except the bottom one, is filled of zeroes except for a one in the position just above the leading diagonal. Equation (35) is a common form of the state equations, used in control system theory and known as the *phase variable* or *companion* form. This form leads to a set of state variables which may not correspond to any physical variables within the system.

The corresponding output relationship is specified by Eq. (31) by noting that $X_i(s) = Z(s)/s^{(n+1-i)}$.

$$y(t) = b_0x_1 + b_1x_2 + b_2x_3 + \cdots + b_{n-1}x_n + b_nz(t). \tag{36}$$

But $z(t) = dx_n/dt$, which is found from the nth state equation in Eq. (34). When substituted into Eq. (36) the output equation is:

$$Y(s) = \begin{bmatrix} \left(b_0 - \frac{b_na_0}{a_n}\right) & \left(b_1 - \frac{b_na_1}{a_n}\right) & \cdots & \left(b_{n-1} - \frac{b_na_{n-1}}{a_n}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \frac{b_n}{a_n}u(t). \tag{37}$$

■ Example 6

Draw a direct form realization of a block diagram, and write the state equations in phase variable form, for a system with the differential equation

$$\frac{d^3y}{dt^3} + 7\frac{d^2y}{dt^2} + 19\frac{dy}{dt} + 13y = 13\frac{du}{dt} + 26u \tag{i}$$

Solution: The system order is 3, and using the structure shown in Fig. 5 the block diagram is as shown in Fig. 6.

The state and output equations are found directly from Eqs. (35) and (37):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -13 & -19 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{ii}$$

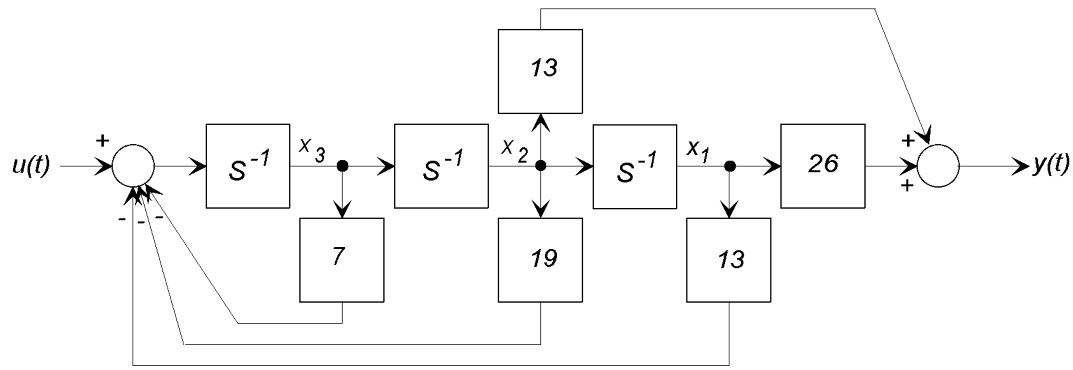

Figure 6: Block diagram of the transfer operator of a third-order system found by a direct realization.

$$y(t) = \left[\begin{array}{ccc} 26 & 13 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] + [0]\, u\,(t)\,. \qquad \text{(iii)}$$

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5 The Matrix Transfer Function

For a multiple-input multiple-output system Eq. 22 is written in terms of the r component input vector $\mathbf{U}(s)$

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\mathbf{U}(s) \qquad (38)$$

generating a set of n simultaneous linear equations, where the matrix is $\mathbf{B}$ is $n \times r$. The m component system output vector $\mathbf{Y}(s)$ may be found by substituting this solution for $\mathbf{X}(s)$ into the output equation as in Eq. 25:

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B}\,\{\mathbf{U}(s)\} + \mathbf{D}\,\{\mathbf{U}(s)\} \\ &= \left[\mathbf{C}\,[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}\right]\{\mathbf{U}(s)\} \end{aligned} \qquad (39)$$

and expanding the inverse in terms of the determinant and the adjoint matrix

$$\begin{aligned} \mathbf{Y}(s) &= \frac{\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \det\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{D}}{\det\,[s\mathbf{I} - \mathbf{A}]}\mathbf{U}(s) \\ &= \mathbf{H}(s)\mathbf{U}(s), \end{aligned} \qquad (40)$$

where $\mathbf{H}(s)$ is defined to be the *matrix transfer function* relating the output vector $\mathbf{Y}(s)$ to the input vector $\mathbf{U}(s)$:

$$\mathbf{H}(s) = \frac{(\mathbf{C}\,\mathrm{adj}\,(s\mathbf{I} - \mathbf{A})\,\mathbf{B} + \det\,[s\mathbf{I} - \mathbf{A}]\,\mathbf{D})}{\det\,[s\mathbf{I} - \mathbf{A}]} \qquad (41)$$

For a system with r inputs $U_1(s), \ldots, U_r(s)$ and m outputs $Y_1(s), \ldots, Y_m(s)$, $\mathbf{H}(s)$ is a $m \times r$ matrix whose elements are individual scalar transfer functions relating a given component of the output $\mathbf{Y}(s)$ to a component of the input $\mathbf{U}(s)$. Expansion of Eq. 41 generates a set of equations:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_m(s) \end{bmatrix} = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1r}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ H_{m1}(s) & H_{m2}(s) & \cdots & H_{mr}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_r(s) \end{bmatrix} \tag{42}$$

where the ith component of the output vector $\mathbf{Y}(s)$ is:

$$Y_i(s) = H_{i1}(s)U_1(s) + H_{i2}(s)U_2(s) + \cdots + H_{ir}(s)U_s(s). \tag{43}$$

The elemental transfer function $H_{ij}(s)$ is the scalar transfer function between the ith output component and the jth input component. Equation 41 shows that all of the $H_{ij}(s)$ transfer functions in $\mathbf{H}(s)$ have the same denominator factor $\det[s\mathbf{I} - \mathbf{A}]$, giving the important result that all input-output differential equations for a system have the same characteristic polynomial, or alternatively have the same coefficients on the left-hand side.

If the system has a single-input and a single-output, $\mathbf{H}(s)$ is a scalar, and the procedure generates the input/output transfer operator directly.