Boundary Layers
Two-Dimensional Steady Boundary Layer Equations

$x$ is horizontal direction along direction of main flow velocity $u$. Velocity at outer edge of boundary layer is called $U_\infty$ or $V_\infty$ or $U_e$ or $V_e$.

$y$ is perpendicular to wall and velocity in this direction is $v$.

The boundary layer begins, say, at $x = 0$ and the boundary layer thickness is $\delta$. $\delta \ll x$. Because the boundary layer is thin, to leading order the pressure is constant through the thickness of the boundary layer, $\frac{\partial P}{\partial y} = 0$. Also, $v \ll u$, and $\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y}$.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}
\]
Boundary Layer Parameters

Thickness of Boundary Layer defined as location where \( u \) is 99% of \( U_e \).

\[
\delta = y \big|_{u/U_e=0.99}
\]

The wall shear stress \( \tau_w \) is given by:

\[
\tau_w = \mu \left( \frac{\partial u}{\partial y} \right)_{\text{wall}} = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}
\]

The skin friction coefficient, \( C_f \), is:

\[
C_f = \tau_w / \left( \frac{1}{2} \rho U_e^2 \right) = \frac{2\tau_w}{\rho U_e^2} = \frac{2\nu}{U_e^2} \left( \frac{\partial u}{\partial y} \right)_{y=0}
\]

The displacement thickness, \( \delta^* \) is the thickness of a flow of speed \( U_e \) that carries a flow rate equal to the deficit in the boundary layer because its speed is less than \( U_e \).

\[
U_e \delta^* = \int_0^\delta (U_e - u) \, dy \quad \delta^* = \int_0^\delta \left( 1 - \frac{u}{U_e} \right) \, dy
\]
Mass Fluxes

\[ \dot{m}_{\text{left}} = \int_0^Y \rho u \, dy \]
\[ \dot{m}_{\text{right}} = \int_0^Y \rho u \, dy + \frac{d}{dx} \left( \int_0^Y \rho u \, dy \right) \delta x \]
\[ \dot{m}_{\text{top}} = -\frac{d}{dx} \left( \int_0^Y \rho u \, dy \right) \delta x \]

Momentum Equation in x direction

\[ \dot{M}_{\text{right}} + \dot{M}_{\text{top}} + \dot{M}_{\text{left}} = F_{\text{pressure}} + F_{\text{stress}} \]

\[ \dot{M}_{\text{left}} = -\int_0^Y \rho u^2 \, dy \]

\[ \dot{M}_{\text{right}} = \int_0^Y \rho u^2 \, dy + \frac{d}{dx} \left( \int_0^Y \rho u^2 \, dy \right) \delta x \]

\[ \dot{M}_{\text{top}} = \dot{m}_{\text{top}} U_e = -U_e \frac{d}{dx} \left( \int_0^Y \rho u \, dy \right) \delta x \]
\[ F_{\text{pressure}} = -\frac{dp}{dx} Y \delta x = \rho U_e \frac{dU_e}{dx} Y \delta x \quad \quad F_{S} = -\tau_w \delta x \]

One additional needed equation is:

\[ Y = \int_{0}^{Y} dy \]

Then all the equations on the last two pages can be combined into:

\[ \frac{d}{dx} \int_{0}^{Y} u(U_e - u)dy + \frac{dU_e}{dx} \int_{0}^{Y} (U_e - u)dy = \frac{\tau_w}{\rho} \]

For \( y > \delta \) the integrands are zero so the upper limits can be changed to \( \delta \).

\[ \frac{d}{dx} \int_{0}^{\delta} u(U_e - u)dy + \frac{dU_e}{dx} \int_{0}^{\delta} (U_e - u)dy = \frac{\tau_w}{\rho} \]

This is Von Karman’s Momentum Integral Equation. It relates the integrals of the velocity profile in the boundary layer to the shear stress and \( U_e \) and \( U_e^2 \) whose x-derivative is proportional to the pressure gradient.

The momentum thickness \( \Theta \) is defined as:

\[ \Theta = \int_{0}^{\delta} \frac{u}{U_e} \left(1 - \frac{u}{U_e}\right) dy \]

With this definition, the momentum integral equation can be written in the following two forms:

\[ \frac{d}{dx} [U_e^2 \Theta] + \delta^* U_e \frac{dU_e}{dx} = \frac{\tau_w}{\rho} \]

\[ \frac{d\Theta}{dx} + (2 + H) \frac{\Theta}{U_e} \frac{dU_e}{dx} = \frac{C_f}{2} \quad \quad \text{where: } H \equiv \frac{\delta^*}{\Theta} \]
A second boundary layer equation comes from equating the kinetic energy change along \( x \) in the boundary layer to the energy input or output from the pressure distribution and the energy dissipation due to shear stresses in the boundary layer.

The kinetic energy thickness, \( \theta^* \), is defined as:

\[
\theta^* = \int_0^\delta \frac{u}{U_e} \left(1 - \frac{u^2}{U_e^2}\right) dy
\]

The kinetic energy dissipation coefficient, \( C_D \), is defined as:

\[
C_D = \frac{D}{\rho u_e^3}
\]

where \( D \) is the dissipation per unit area (along and perpendicular to the surface).

Using these definitions, the kinetic energy equation is:

\[
\frac{d\theta^*}{dx} + 3 \frac{\theta^*}{u_e} \frac{du_e}{dx} = 2C_D
\]

The energy thickness ratio, \( H^* \), is defined as:

\[
H^* = \frac{\theta^*}{\theta}
\]

It is common to combine the kinetic energy equation and Von Karman's momentum equation to obtain:

\[
\frac{\theta}{H^*} \frac{dH^*}{dx} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H - 1) \frac{\theta}{u_e} \frac{du_e}{dx}
\]
Example of Solution of Momentum Integral BL Equation

\[ U_e = 2m/s \quad \delta(x) = 0.01(1-e^{-0.1x}) \quad \frac{u(y)}{U_e} = (1 - e^{-k(x)}y)^2 \quad \rho = 1000kg/m^3 \]

Problem: Determine the shear stress, \( \tau \), at \( x = 5 \) meters.

Determination of \( k\delta \) from BL thickness:

\[ 0.99 = (1 - e^{-k(x)\delta(x)})^2 \quad \rightarrow \quad k(x)\delta(x) = 5.3 \quad k(x) = \frac{5.3}{\delta(x)} \]

At \( x = 5 \) m, \( k = 1347 \) m\(^{-1}\).

\[
\frac{d}{dx} \int_0^{0.01[1-\exp(-0.1x)]} U_e(1 - e^{k(x)y})^2 [U_e - U_e(1 - e^{-ky})^2] \, dy + 0 = \frac{\tau}{\rho}
\]

\[ 0.01(1 - e^{-0.1 \times 5}) = 0.01(1 - e^{-0.5}) = 0.00393 \]

\[
\frac{\tau}{\rho} = U_e^2(1 - e^{-5.3})^2 [1 - (1 - e^{-5.3})^2] \frac{d}{dx} [0.01(1 - e^{-0.1x})] \\
+ \int_0^{0.00393} \frac{d}{dx} \{U_e(1 - e^{-k(x)y})^2 [U_e - U_e(1 - e^{-k(x)y})^2]\} \, dy \\
= U_e^2(0.000060 + 0.000100) = 0.00016 U_e^2
\]

\[ \tau = 1000 \times 4 \times 0.00016 = 0.64N/m^2 \]

\[ c_f = \frac{\tau}{\frac{1}{2} \rho U_e^2} = 0.00032 \]

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Calculation of Turbulent Boundary Layer
when Pressure Distribution is Known

This result is approximate since the boundary layer thickness will alter the pressure distribution.

The principal unknowns (quantities to be determined) are: $\theta(x)$ and $\delta^*(x)$. An equivalent set of unknowns is $\theta(x)$ and $H(x)$.

There are two fundamental equations:

$$\frac{d\theta}{dx} = -(H + 2)\frac{\theta}{U_e} \frac{dU_e}{dx} + \frac{C_f}{2} \tag{1}$$

$$\frac{\theta \frac{dH^*}{H^*}}{dx} = \frac{2C_D}{H^*} - \frac{C_f}{2} + (H - 1)\frac{\theta}{u_e} \frac{du_e}{dx} \tag{2}$$

To be able to integrate the unknowns along the boundary layer, the derivatives of each of them are required: $d\theta/dx$ and $dH/dx$. Equation 1 is in the desired form. To put equation 2 in the desired form, use the chain rule:

$$\frac{dH^*}{dx} = \frac{dH}{dx} \frac{dH^*}{dH} \tag{3}$$

Empirical “closure relations” for $H^*(H)$ and $dH^*/dH$ exist. Therefore we write the energy equation in the desired form as:

$$\frac{dH}{dx} = \frac{H^*}{\theta} \frac{1}{dH^*/dH} \left[ \frac{2C_D}{H^*} - \frac{C_f}{2} + (H - 1)\frac{\theta}{u_e} \frac{du_e}{dx} \right] \tag{4}$$
To do the integrals numerically, we need a means of determining $C_f$, $C_D$, $H^*$ and $dH^*/dH$ in terms of the principal quantities $H$ and $R_\theta$, where $R_\theta = U_e\theta/\nu$. These empirical "closure relations" have been determined by assembling a large amount of experimental data.

**Laminar Closure Relations**

$$H^* = \begin{cases} 
0.76(H - 4)^2/H + 1.515, & H < 4.0 \\
0.015(H - 4)^2/H + 1.515, & H \geq 4.0
\end{cases}$$

$$C_f = \begin{cases} 
[0.03954[(7.4 - H)^2/(H - 1.0)] - 0.134]/R_\theta, & H < 7.4 \\
0.044[1.0 - 1.4/(H - 6)]^2 - 0.134)]/R_\theta, & H \geq 7.4
\end{cases}$$

$$\frac{2C_D}{H^*} = \begin{cases} 
0.00205(4 - H)^{5.5} + 0.207]/R_\theta, & H < 4.0 \\
-0.003(H - 4.0)^2/(1 + 0.02(H - 4)^2 + 0.207]/R_\theta, & H \geq 4.0
\end{cases}$$

**Turbulent Closure Relations**

$$H_o = \begin{cases} 
3 + 400/R_\theta, & R_\theta > 400 \\
4, & R_\theta \leq 400
\end{cases}$$

$$R_{\theta z} = \begin{cases} 
R_\theta, & R_\theta > 200 \\
200, & R_\theta \leq 200
\end{cases}$$

$$H^* = \begin{cases} 
1.505 + 4/R_\theta + (0.165 - 1.6/\sqrt{R_\theta})(H_o - H)^{1.6}/H, & H < H_o \\
(H - H_o)^2[0.007\ln(R_\theta)/[H - H_o + 4/\ln(R_\theta)]^2 + 0.015/H] + 1.505 + 4.0/R_\theta, & H \geq H_o
\end{cases}$$

$$C_f = 0.3e^{-1.33H}\left[\frac{\ln(R_\theta)}{2.3026}\right]^{-1.74 + 0.31H}$$

$$\frac{2C_D}{H^*} = 0.5C_f\frac{4.0/H - 1}{3} + 0.03\left(1 - \frac{1}{H}\right)^3$$
\begin{align*}
\text{Initial condition (or boundary condition)}
\end{align*}

At \( x = 0.01 \), \( \theta = 0.0003 \), \( \theta = 0.0003 \)

\( h = 1.4 \)