Sea Waves

Dominated by inviscid irrotational solution \((\nabla^2 \phi = 0)\)

**Free Surface Boundary Conditions**

\[
\frac{\partial \Phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \left[ \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \zeta}{\partial y} \right]_{z=\zeta} \quad \text{(kinematic)}
\]

\[
\left\{ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 \right] \right\}_{z=\zeta} + g\zeta = \text{constant} \quad \text{(dynamic)}
\]

**Linearized Boundary Conditions**

(linear terms in perturbation potential dominate higher order terms)

Case of onset flow velocity of \(-iU\).
Now \(\phi\) is the perturbation potential and \(\Phi = -Ux + \phi\).

\[
\left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = \frac{\partial \zeta}{\partial t} - U \frac{\partial \zeta}{\partial x} \quad \left[ \frac{\partial \phi}{\partial t} - U \frac{\partial \phi}{\partial x} \right]_{z=0} + g\zeta = 0
\]

For steady flow with onset flow:

\[
\left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = -U \frac{\partial \zeta}{\partial x} \quad U \left[ \frac{\partial \phi}{\partial x} \right]_{z=0} = g\zeta \quad \left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = -\frac{U^2}{g} \frac{\partial^2 \phi}{\partial x^2}
\]

Case of 2D waves and zero onset flow so \(\phi\) is the total potential.

\[
\left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = \frac{\partial \zeta}{\partial t} \quad \left[ \frac{\partial \phi}{\partial t} \right]_{z=0} + g\zeta = 0
\]

\[
\frac{\partial^2 \phi}{\partial t^2} \bigg|_{z=0} = -g \frac{\partial \zeta}{\partial t} \quad \left[ \frac{\partial \phi}{\partial z} \right]_{z=0} = -\frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \bigg|_{z=0}
\]
\[ \frac{\partial^2 \phi}{\partial t^2} \bigg|_{z=0} = -g \frac{\partial \zeta}{\partial t} \quad \frac{\partial \phi}{\partial z} \bigg|_{z=0} = - \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \bigg|_{z=0} \quad \zeta = \frac{-1}{g} \frac{\partial \phi}{\partial t} \bigg|_{z=0} \]

**Deep Water**

\[ \zeta = Ae^{i(kx - \omega t)} \quad \phi = B e^{kz} e^{i(kx - \omega t)} \quad \text{Traveling wave satisfying Laplace's Equation} \]

\[ Bke^{kz} e^{i(kx - \omega t)} = \frac{1}{g} \omega^2 Be^{kz} e^{i(kx - \omega t)} \quad k = \frac{\omega^2}{g} \quad \omega^2 = kg \]

\[ \zeta = \frac{-1}{g} (-i\omega) B e^{i(kx - \omega t)} \]

\[ A = \frac{i\omega}{g} B \quad B = \frac{-i\omega}{\omega} A = -\frac{i}{k} A \]

**Finite Depth**

\[ \zeta = Ae^{i(kx - \omega t)} \quad \phi = B \cosh k(z + h) e^{i(kx - \omega t)} \]

\[ Bk \sinh kh e^{i(kx - \omega t)} = B \omega^2 \frac{1}{g} \cosh kh e^{i(kx - \omega t)} \]

\[ k \tanh kh = \frac{\omega^2}{g} \quad \omega^2 = gk \tanh kh \]

\[ \zeta = \frac{-1}{g} (-i\omega) B \cosh(kh) e^{i(kx - \omega t)} = \frac{i\omega}{g} B \cosh(kh) e^{i(kx - \omega t)} \quad A = \frac{i\omega}{g} \cosh(kh) B \]

**Dispersion Relations** for waves of circular frequency \( \omega = 2\pi f \) and wavenumber \( k = 2\pi / \lambda \) and zero onset flow.

\[ \omega^2 = gk \quad \text{deep water} \]

\[ \omega^2 = gk \tanh kh \quad \text{water of depth } h \]
Generation of Random Wave Form From Sinusoidal Components

All curves have zero mean. Individual wave contributions shown vertically displaced for viewing clarity.

Heavy line is the sum of the individual wave contributions.
Example of Simulation

Suppose a two dimensional (long crested) wave is generated with a wave-maker in a wave tank with an elevation at a specified location given by $z(t)$, where:

$$z(t) = 0.97 \sin(5.2t + 0.82) + 0.99 \sin(7.8t + 1.24) + 1.08 \sin(9.8t + 2.72)$$

What is the maximum elevation that occurs in the time interval of 0 to 120 seconds (2 minutes). The usual way of finding maxima of analytic functions by setting the derivative to zero is not practical here because there are a great many maxima and the largest of these must be determined. However, because of the great computational speed of common computers, this can be done numerically without much effort.

```matlab
% MATLAB Version of program Sinmax
% t = 0:0.01:120;
% z = 0.97*sin(5.2*t + 0.82) ...
% + 0.99*sin(7.8*t +1.24) + 1.08*sin(9.8*t + 2.72);
% zmax = max(z);
% mmax = find(z == zmax);
% tmax = (t(mmax));
% fprintf (1,'tmax = %7.3f zmax = %8.4f
',tmax,zmax);
% q = [t;z];
% fid = fopen('zmaxm.dat','w');
% fprintf(fid,'%f7.2f %8.4f
',q);
% plot(t,z);
% xlabel('t');
% ylabel('z');
% title('Sinmaxm');

>> sinmaxm
  tmax = 118.490  zmax =  2.9447
>>
```
Sea Spectra

We consider wave fields whose statistics are both stationary and homogeneous in the horizontal plane.

A sea spectrum function $S_T(k, \omega, \theta)$ is a partial description of the statistics of the wave field defined such that $S_T(k, \omega, \theta) \delta k \delta \omega \delta \theta$ is the contribution to the average wave energy per unit surface area, $E$, in the wavenumber, wave circular frequency and propagation angle bands; $\delta k \delta \omega \delta \theta$.

For surface elevation $\zeta(x, t)$ the average wave energy is defined as:

$$E = \langle \zeta^2 \rangle$$

where $\langle \rangle$ signifies the statistical, temporal or spatial average.

Thus:

$$\langle \zeta^2 \rangle = \int_0^{2\pi} \int_0^\infty \int_0^\infty S_T(k, \omega, \theta) \delta k \delta \omega \delta \theta$$

Similar definitions apply when frequency, $f$, is used instead of circular frequency, $\omega$, and/or when spatial frequency, $\frac{1}{\lambda}$, is used instead of wavenumber, $k$.

For the frequently encountered case of linear, deep water gravity waves the circular frequency and the wavenumber are related to each other through the dispersion relation

$$\omega^2 = gk$$

so that $\omega$ and $k$ are not independent of each other. Then the spectrum is a function of only one or the other of these variables and can be written as: $S_t(\omega, \theta)$ or $S_x(k, \theta)$. These functions are related by:

$$S_x(k, \theta) = \frac{g}{2\omega} S_t(\omega, \theta)$$

Hence:

$$\langle \zeta^2 \rangle = \int_0^{2\pi} \int_0^\infty S_x(k, \theta) dk d\theta = \int_0^{2\pi} \int_0^\infty S_x(\omega, \theta) d\omega d\theta$$

For unidirectional (long crested) seas, all the waves are in a single direction and the spectra are described by $S_t(\omega)$ or $S_x(k)$.

$$\langle \zeta^2 \rangle = \int_0^\infty S_t(\omega) d\omega = \int_0^\infty S_x(k) dk$$
The fundamental linearized plane progressive wave is:

\[ \zeta = Ae^{ikx - \omega t} \]

\[ \phi = -\frac{i\omega A}{k} e^{kz} e^{ikx - \omega t} \]

Random sea waves have spectrum \( S(\omega, \theta) \).
For the 2D case the spectrum is \( S(\omega) \).

\[ \int_{\omega_1}^{\omega_2} S(\omega) \, d\omega \]  

is the contribution to \( \overline{\zeta^2} \) of waves with circular frequencies between \( \omega_1 \) and \( \omega_2 \).