Wave Statistics

One way to calculate wave statistics is directly from long-term simulations.

Example What is the expected value of the largest wave elevation in a day?

Solution by simulation from a known wave spectrum.

1. Simulate waves for many days.
2. List the largest elevation in each day.
3. Calculate the average of the values in the list.

Another Example What is the probability that the largest wave elevation in one day is less than the value V. Solution by simulation.

1. Simulate waves for many days.
2. Determine the fraction of days that the elevation does not exceed V.
3. This fraction is an estimate of the desired probability.

The above direct approach is cumbersome and computationally intensive. Many wave statistics have been theoretically determined in terms of the wave spectrum. The associated formulae can be determined using numerical integration.
Results from Theory

The spectral moments, \( m_n \), are defined in terms of the one-sided spectrum, \( S_W(\omega) \), as:

\[
m_n = \int_0^\infty \omega^n S_W(\omega) d\omega
\]

The following results apply when the surface elevation is a gaussian random process.

Number of Waves per Unit Time

The average number of times the wave elevation, \( \zeta \), crosses the mean sea level (\( \zeta = 0 \)) per unit time while increasing is called \( f_o \) and given by:

\[
f_o = \frac{1}{2\pi} \sqrt{\frac{m_2}{m_0}}
\]

The average number of wave crests per unit time is called \( f_c \) and is given by:

\[
f_c = \frac{1}{2\pi} \sqrt{\frac{m_4}{m_2}}
\]

The bandwidth, \( \epsilon \), is given by:

\[
\epsilon = \sqrt{1 - \frac{f_o^2}{f_c^2}}
\]
Definition of a gaussian random process For any number of variables, the joint probability density (pdf) of all the variables is a joint gaussian random variable at each time for a gaussian random process. This probability density function is given by:

\[
p(x_1, x_2, ..., x_n) = \frac{1}{\sqrt{(2\pi)^n|\Delta|}} \exp\left\{-\frac{1}{2} [X]^T [\Delta^{-1}] [X]\right\}
\]

\([X]\) is the column vector of the variables. \(\Delta\) is the n-by-n covariance matrix whose elements are given by:

\[
\Delta_{ij} = E[x_i x_j]
\]

For most wave statistics of interest, the doubly joint pdf between surface elevation, \(\zeta\) and vertical surface velocity, \(\dot{\zeta}\), and the triply joint pdf where the surface acceleration, \(\ddot{\zeta}\), is included are all that are needed.

\[
p(\zeta, \dot{\zeta}) = \frac{1}{2\pi \sqrt{m_0 m_2}} \exp\left[-\frac{m_2 \zeta^2 + m_0 \dot{\zeta}^2}{2m_0 m_2}\right]
\]

\[
p(\zeta, \dot{\zeta}, \ddot{\zeta}) = \frac{1}{(2\pi)^{3/2} \sqrt{m_0 m_4 - m_2^2}} \exp\left[-\frac{m_2 m_4 \zeta^2 + (m_0 m_4 - m_2^2) \dot{\zeta}^2 + m_0 m_2 \ddot{\zeta}^2 + 2m_2^2 \zeta \ddot{\zeta}}{2m_2 (m_0 m_4 - m_2^2)}\right]
\]
The normalized Gaussian probability distribution function (pdf), $\Psi(x)$, is:

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}dz$$

Call the crest height $\xi$.

The normalized crest height, $\eta$, is defined by: $\eta = \frac{\xi}{\sqrt{m_0}}$

The probability distribution function for $\eta$ is:

$$P(\eta) = \Psi \left( \frac{\eta}{\epsilon} \right) - \sqrt{1 - \epsilon^2} e^{-\eta^2/2} \Psi \left( \frac{\sqrt{1 - \epsilon^2}}{\epsilon} \eta \right)$$

and the pdf for $\eta$ is:

$$p(\eta) = \frac{\epsilon}{2\pi} \exp \left[ -\frac{\eta^2}{2\epsilon^2} \right] + \sqrt{1 - \epsilon^2} \eta e^{-\eta^2/2} \Psi \left( \frac{\sqrt{1 - \epsilon^2}}{\epsilon} \eta \right)$$

![Figure 25](image.png)

**Figure 25** Probability density function of $\eta$ for various values of the band width $\epsilon$.

Typically, $\epsilon \approx 0.6$.

For engineering purposes we are interested in large seas ($\eta >> 1$). This corresponds to the tail of the pdf for $\eta$. In this region:

$$p(\eta) = \sqrt{1 - \epsilon^2} \eta e^{-\eta^2/2} \quad P(\eta) = 1 - \sqrt{1 - \epsilon^2} e^{-\eta^2/2}$$
Average Amplitude of the $1/n$th Highest waves

Call the smallest normalized wave amplitude in the $1/n$th highest Waves $\eta_{1/n}$:
\[
\frac{1}{n} = 1 - P(\eta_{1/n})
\]

Example: $n = 10$.
1 - (probability that a wave amplitude is less than the smallest of the 10% largest waves) is $1/10$.
This is because the probability that a (random) wave is smaller than 10% is 90%.

For $n >> 1$, use the approximate $P$.
\[
\frac{1}{n} = \sqrt{1 - \epsilon^2} \exp \left[ -\frac{1}{2} \eta_{1/n}^2 \right]
\]
\[
\eta_{1/n} = \sqrt{2 \ln(n \sqrt{1 - \epsilon^2})}
\]

Amongst the $1/n$th highest waves, the conditional pdf is:
\[
p_{\eta > \eta_{1/n}}(\eta) = np(\eta) = n\sqrt{1 - \epsilon^2} \eta \exp(-\eta^2/2), \quad \eta_{1/n} < \eta < \infty
\]

The expectation of these amplitudes is the average of the $1/n$th highest waves.
\[
\overline{\eta_{1/n}} = n\sqrt{1 - \epsilon^2} \int_{\eta_{1/n}}^{\infty} \eta^2 e^{-\eta^2/2} d\eta
\]

Let $n' = \sqrt{1 - \epsilon^2} n$. Then, $n'$ is the number of zero up-crossings in a record with $n$ crests. The result of the integration is:
\[
\overline{\eta_{1/n}} = n' \left( \frac{\sqrt{2 \ln n'}}{n'} + \sqrt{2\pi} \left[ 1 - \Psi\left(\sqrt{2 \ln n'}\right) \right] \right)
\]
Extreme Waves

Consider \( n \) non-dimensional random wave Amplitudes. Each has same pdf.

What are the probabilities of the largest waves in the set?

Approach
Order the waves from smallest to largest.
\( \phi_1 \) is the smallest and \( \phi_n \) is the largest wave amplitude. Now, each of the \( \phi \)'s has a different pdf.
We want to find the pdf for \( \phi_n \).

Probability that \( \phi_n \) is less than a particular value \( \phi_{n_o} \) is equal to the probability that all the waves are smaller than \( \phi_{n_o} \).

\[
P_{\phi_n}(\phi_{n_o}) = [P_\eta(\phi_{n_o})]^n
\]

The amplitude that has a probability, \( \alpha \), of being exceeded by \( \phi_n \) is called \( \alpha \phi_n \).

\[
P_{\phi_n}(\alpha \phi_n) = [P_\eta(\alpha \phi_n)]^n = 1 - \alpha
\]

Meaning of the Nomenclature
Suppose \( \alpha = 0.01 \). Then the amplitude whose probability of being exceeded by \( \phi_n \) is 0.01 is named \( \phi_{0.01} \).
The probability that \( \phi_n \) is less than \( \phi_{0.01} \) is 0.99.

\[
P_\eta(\alpha \phi_n) = (1 - \alpha)^{1/n}
\]

\[
\Psi \left( \frac{\alpha \phi_n}{\epsilon} \right) - \sqrt{1 - \epsilon^2} \exp \left[ -\frac{1}{2} \frac{\alpha \phi_n^2}{\epsilon^2} \right] \Psi \left( \frac{\sqrt{1 - \epsilon^2}}{\epsilon} \alpha \phi_n \right) = (1 - \alpha)^{1/n}
\]
Since we are interested large waves, we can use the expressions for the tails of the probability functions:

$$P(\eta) = 1 - \sqrt{1 - \epsilon^2} e^{-\eta^2/2}$$

Then, $1 - \sqrt{1 - \epsilon^2} \exp \left[ -\frac{1}{2} \alpha \phi_n^2 \right] = (1 - \alpha)^{1/n}$

Solve for $\alpha \phi_n$:

$$\alpha \phi_n = \sqrt{2 \ln \left( \frac{\sqrt{1 - \epsilon^2}}{1 - (1 - \alpha)^{1/n}} \right)}$$

Note: The value of $n$ for a given period of time $T$ can be obtained from:

$$f_c = \frac{1}{2\pi} \sqrt{\frac{m_4}{m_2}}$$

$$n = f_c T = \frac{T}{2\pi} \sqrt{\frac{m_4}{m_2}}$$