22.51 Problem Set 4 (due Fri, Oct. 5)

1 Oscillation (Sturm) Theorem (25 pt)

**Question**: Suppose function $\psi_1(x)$ satisfies,

$$
\psi_1''(x) = \frac{2m}{\hbar^2} (V(x) - E_1) \psi_1(x), \quad x \in (-\infty, \infty),
$$

and function $\psi_2(x)$ satisfies,

$$
\psi_2''(x) = \frac{2m}{\hbar^2} (V(x) - E_2) \psi_2(x), \quad x \in (-\infty, \infty),
$$

where all relevant quantities are real.

(a). Prove that,

$$
\psi_1'(x)\psi_2(x) - \psi_1(x)\psi_2'(x)|_{x_0}^{x_1} = \frac{2m}{\hbar^2} (E_2 - E_1) \int_{x_0}^{x_1} \psi_1(x)\psi_2(x)dx,
$$

for any $x_1 > x_0$.

(b). Suppose $E_1 < E_2$, prove that between any two consecutive nodes of $\psi_1(x)$, there exists at least one node of $\psi_2(x)$.

(c). When both $\psi_1(x)$ and $\psi_2(x)$ are bound-states and $V(x)$ is smooth, and $E_1 < E_2$, show that there exist extra nodes of $\psi_2(x)$ to the left of the leftmost node of $\psi_1(x)$, and extra nodes to the right of the rightmost node of $\psi_1(x)$. Therefore, $\psi_2(x)$ must have at least one more node than $\psi_1(x)$.

2 Bloch Theorem (25 pt)

**Question**: The Hamiltonian of a single particle in a 1D periodic potential is,

$$
\hat{\mathcal{H}} \equiv \frac{\hat{p}^2}{2m} + V(x), \quad V(x - a) = V(x) \quad \forall x \in (-\infty, \infty).
$$

Let us define translation operator $\hat{T}$,

$$
\hat{T}\psi(x) \equiv \psi(x - a).
$$
Prove that,

(a). Each eigenfunction $\psi(x)$ of $\hat{T}$ must satisfy,

$$\hat{T}\psi(x) = \exp(-ika)\psi(x),$$  \hspace{1cm} (6)

for some $k \in [-\pi/a, \pi/a)$.

(b). Rationalize without rigorous proof that we should be able label the eigenfunctions of $\hat{H}$ by $k \in [-\pi/a, \pi/a)$ and maybe other quantum number(s) $n$,

$$\hat{T}\psi^0_k(x) = \exp(-ika)\psi^0_k(x), \quad \hat{H}\psi^0_k(x) = E_n(k)\psi^0_k(x),$$  \hspace{1cm} (7)

where $E_n(k)$ is called the energy band.

3 Bound-State(s) of $\delta$-Function Potential

**Question**: Solve for the bound-state(s) of,\[\hat{H} \equiv \hat{p}^2/2m - V_0\delta(x), \quad V_0 > 0.\]  \hspace{1cm} (8)

4 Eigenfunctions of comb-Function Potential

**Question**: Using the conclusion of 2(b), solve for the eigenfunctions of,\[\hat{H} \equiv \hat{p}^2/2m - V_0 \sum_{n=-\infty}^{\infty} \delta(x-n), \quad V_0 > 0.\]  \hspace{1cm} (9)