1. (a) Show that the stress tensor is symmetric in the absence of body forces and accelerations. Assume that the areas of the faces of the unit cube are small enough.

The net moments on an element should be equal to zero. From the Figure above, the total moment is obtained as

\[
\sum M = \left( \sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} \delta x_1 \right) \delta x_2 \delta x_3 \frac{\delta x_1}{2} + \left( \sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2} \delta x_2 \right) \delta x_1 \delta x_3 \frac{\delta x_2}{2} - \left( \sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2} \delta x_2 \right) \delta x_1 \delta x_3 \frac{\delta x_2}{2} - \sigma_{12} \delta x_1 \delta x_3 \frac{\delta x_2}{2}
\]

In static equilibrium, the total moment is zero. As a result, following expression is obtained as

\[
\left( \sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_1} \delta x_1 \right) + \sigma_{21} - \left( \sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_2} \delta x_2 \right) - \sigma_{12} = 0
\]

For a infinitesimal limit, the derivatives become small compared to the shear terms.

\[
\sigma_{21} + \sigma_{21} - \sigma_{12} - \sigma_{12} = 0, \quad \sigma_{21} = \sigma_{12}
\]

The same relation is obtained for \(x_2-x_3\) plane and \(x_3-x_1\) plane.

\[
\sigma_{ji} = \sigma_{ji}
\]

(b) At a point of interest in an engineering component, the stresses with respect to a convenient coordinate system are:

\[
\begin{align*}
\sigma_{11} &= 100, \quad \sigma_{22} = -60, \quad \sigma_{33} = 40 \text{ MPa} \\
\tau_{12} &= 80, \quad \tau_{23} = \tau_{31} = 0 \text{ MPa}
\end{align*}
\]
Write down matrix of stress tensor and calculate the traction vector acting on a plane defined by a unit normal vector \( n_j = \frac{2}{3} \hat{x} - \frac{2}{3} \hat{y} + \frac{1}{3} \hat{z} \).

\[
\sigma_{ij} = \begin{pmatrix}
100 & 80 & 0 \\
80 & -60 & 0 \\
0 & 0 & 40
\end{pmatrix} \text{ MPa, } T_j = \sigma_{ij} n_j
\]

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3
\end{pmatrix} = \begin{pmatrix}
100 & 80 & 0 \\
80 & -60 & 0 \\
0 & 0 & 40
\end{pmatrix} \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 40/3 \\ 280/3 \\ 40/3 \end{pmatrix} \text{ MPa}
\]

(c) Calculate three invariants of stress. \((I_1, I_2, I_3)\)

\[
I_1 = \sigma_{ii} = 100 - 60 + 40 = 80 \text{ MPa}
\]

\[
I_2 = -(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}) + (\sigma_{23}^2 + \sigma_{13}^2 + \sigma_{12}^2) = 10800 \text{ MPa}^2
\]

\[
I_3 = \det(\sigma_{ij}) = -496000 \text{ MPa}^3
\]

2. For the stress state of

\[
\sigma_{ij} = \begin{pmatrix}
5 & 0 & 4 \\
0 & 5 & 3 \\
4 & 3 & 5
\end{pmatrix} \text{ MPa}
\]

(a) Calculate the principal stresses and the components of unit vectors normal to the principal plane for each principal stress.

\[
\begin{vmatrix}
5 - \lambda & 0 & 4 \\
0 & 5 - \lambda & 3 \\
4 & 3 & 5 - \lambda
\end{vmatrix} = 0
\]

\[
(5 - \lambda)[(5 - \lambda)(5 - \lambda) - 9] + 4(5 - \lambda) = 0
\]

\[
(5 - \lambda)\lambda^2 - 10\lambda = 0
\]

\(\lambda_1 = 0 \text{ MPa, } \lambda_2 = 5 \text{ MPa, } \lambda_3 = 10 \text{ MPa}\)

For \(\lambda_1 = 0 \text{ MPa}, \)

\[
\begin{pmatrix}
5 & 0 & 4 \\
0 & 5 & 3 \\
4 & 3 & 5
\end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 0 \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \text{ with } n_1^2 + n_2^2 + n_3^2 = 1.
\]
\[ n_1 = -2\sqrt{2}/5, \; n_2 = -3\sqrt{2}/10, \; n_3 = \sqrt{2}/2 \]

For \( \lambda_2 = 5 \text{ MPa} \),
\[
\begin{pmatrix}
0 & 0 & 4 \\
0 & 5 & 3 \\
4 & 3 & 5
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
= 5 \cdot
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
\text{ with } n_1^2 + n_2^2 + n_3^2 = 1.
\]

For \( \lambda_3 = 10 \text{ MPa} \),
\[
\begin{pmatrix}
5 & 0 & 4 \\
0 & 5 & 3 \\
4 & 3 & 5
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
= 10 \cdot
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
\text{ with } n_1^2 + n_2^2 + n_3^2 = 1.
\]

(b) Calculate the deviatoric stress.

\[ s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \sigma_{ij}, \quad \frac{1}{3} \sigma_{kk} = 5 \text{ MPa} \]
\[ s_{ij} = \begin{pmatrix}
0 & 0 & 4 \\
0 & 0 & 3 \\
4 & 3 & 0
\end{pmatrix} \]

(c) Calculate the second invariant of the deviatoric stress.

\[ J_2 = \frac{1}{2} s_{ij} s_{ij} = \frac{1}{6} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2 \]
\[ = 25 \text{ MPa} \]

3. Determine whether the following stress state is permissible in viewpoint of static equilibrium.

\[ \sigma_{ij} = \begin{pmatrix}
x_2 x_3 & x_1^2 & x_1^2 x_2 \\
x_1^2 & -2x_1 x_2 & x_1 \\
x_1^2 x_2 & x_1 & -2x_1 x_2 x_3
\end{pmatrix} \]

For static equilibrium, (i) \( \sigma_{ij} = \sigma_{ji} \) (total Moment=0)
(ii) $\sigma_{ii,j} = 0$ (total force =0)

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 2x_1 - 2x_i = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 2x_1x_2 + 0 - 2x_1x_2 = 0$$

Yes it is permissible.

4. A solid sphere is subject to a uniform biaxial stress:

$$\sigma_{ij} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The unit normal to the surface is $n_j = (n_1, n_2, n_3)$. What is the normal traction on the surface?

$$T_i = \sigma_{ij}n_j = (\sigma n_1, -\sigma n_2, 0), \quad T^n = \sigma_{ij}n_j n_i = (\sigma n_1, -\sigma n_2, 0)\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \sigma n_1^2 - \sigma n_2^2$$

For spherical coordinate, $n_j = (\sin \theta \cdot \cos \phi, \sin \theta \cdot \sin \phi, \cos \theta)$,

$$T^n = \sigma \sin^2 \theta (\cos^2 \phi - \sin^2 \phi)$$

5. An element in plane stress at the surface of a large machine is subjected to stresses

$\sigma_x = 150MPa$, $\sigma_y = -50MPa$, and $\tau_{xy} = 40MPa$ as shown in the figure.
By constructing Mohr’s circle, determine
(a) the principal stresses.

\[ \sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} = 50 \text{ MPa}, \quad R = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2} = \sqrt{100^2 + 40^2} = 107.7 \text{ MPa} \]

\[ \sigma_1 = \sigma_{\text{avg}} + R = 157.7 \text{ MPa}, \quad \sigma_2 = \sigma_{\text{avg}} - R = -57.7 \text{ MPa} \]

(b) the maximum shear stresses

The orientation of the principal stress with respect to the reference frame is:

For \( \sigma_1 \), \( \tan 2\theta_1 = \frac{40}{100} = 0.4 \) \( \theta_1 = 10.9^\circ \)

\( \sigma_2 \) is oriented 90 degrees from \( \sigma_1 \) and thus \( 10.9 + 90 = 100.9^\circ \).

(b) the maximum shear stresses

The maximum shear stresses are equivalent to to R in magnitude and located

(c) the stresses acting on an element inclined at an angle \( \theta \).

The stresses on an element inclined at \( \theta = 40^\circ \) can be found by determining the x and y components of the line segments AB or BC. The point C is \( 80^\circ - 2(10.9^\circ) = 58.2^\circ \)

Point C: \( \sigma_{x1} = 50 + 107.7(\cos58.2^\circ) = 106.75 \text{ MPa} \), \( -\tau_{x1} = 107.7(\sin58.2^\circ) = +91.5 \text{ MPa} \)

Point A: \( \sigma_{y1} = 50 - 107.7(\cos58.2^\circ) = -6.75 \text{ MPa} \), \( \tau_{y1} = -107.7(\sin58.2^\circ) = -91.5 \text{ MPa} \)
6. Deformation tensor is given as follows.

\[ D_{ij} = \begin{pmatrix} 8 & -1 & -1 \\ 1 & 6 & 0 \\ -5 & 0 & 2 \end{pmatrix} \times 10^{-6} \]

(a) Calculate strain tensor \( (\varepsilon_{ij}) \) and rotational tensor \( (\omega_{ij}) \)

\[ \omega_{ij} = \frac{1}{2} (D_{ij} - D_{ji}) = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \times 10^{-6} \; \text{anti-symmetric} \]

\[ \varepsilon_{ij} = \frac{1}{2} (D_{ij} + D_{ji}) = \begin{pmatrix} 8 & 0 & -3 \\ 0 & 6 & 0 \\ -3 & 0 & 2 \end{pmatrix} \times 10^{-6} \; \text{Symmetric} \]

(b) Calculate the dilatation caused by the strain tensor in (a).

\[ \text{Dilatation} = \frac{\Delta V}{V} = \varepsilon_{ii} = 16 \times 10^{-6} \]

(c) Calculate the principal strains.

\[ \begin{vmatrix} 8 - \lambda & 0 & -3 \\ 0 & 6 - \lambda & 0 \\ -3 & 0 & 2 - \lambda \end{vmatrix} = 0 \]

\[ (8 - \lambda)[(6 - \lambda)(2 - \lambda)] - 3 \cdot 3(6 - \lambda) = 0 \]

\[ \lambda_1 = 6 \]

\[ \lambda_{2,3} = 5 \pm 3 \sqrt{2} \].

By multiplying the ignored factor of \( 10^{-6} \)

\[ \lambda_1 = 6 \times 10^{-6}, \; \lambda_2 = 9.24 \times 10^{-6}, \; \lambda_3 = 0.76 \times 10^{-6}. \]