3.22 Mechanical Behavior of materials
PS4 Solution
Due: March, 9, 2004 (Tuesday) before class (10:00am)

1. For the case of an epoxy + 70 v/o (volume percent) S-glass long fiber composite, what is the elastic modulus of the composite both parallel and perpendicular to the axis of the fibers? Assume $E_{\text{epoxy}} = 3$ GPa and $E_{\text{glass}} = 85$ GPa.

For the case of loading parallel to the fibers,

$$E_{\parallel} = E_f V_f + E_m V_m \quad \text{so that} \quad E_{\parallel} = 85 \times 10^9 (0.7) + 3 \times 10^9 (0.3) = 60.4 \text{GPa}.$$  

For the case of loading perpendicular to the fibers,

$$E_{\perp} = \frac{E_f E_m}{V_f E_m + (1-V_f)E_f} = \frac{85 \times 10^9 (3 \times 10^9)}{0.7(3 \times 10^9) + 0.3(85 \times 10^9)} = 9.24 \text{GPa}.$$  

2. This problem is to estimate the modulus of a particulate composite using a “cube in a cube” model as shown below. Cube 1 and 2 have Young’s moduli $E_1$ and $E_2$, respectively.

(a) Show that the modulus of the composite shown above is:

$$E_c = E_2 \left(1 - \left(\frac{l_1}{l_2}\right)^2\right) + \frac{E_1 E_2}{E_1 (1 - l_1/l_2) + E_2 (l_1/l_2) \left(\frac{l_1}{l_2}\right)^2}.$$  

Cross section near the center
First, for the volume above, we can estimate the modulus $E_c'$. This volume element is in an isostress state with respect to the applied stress, and as a result,

$$E_c' = \frac{E_1 E_2}{E_1 V_2' + E_2 V_1'} = \frac{E_1 E_2}{E_1 \left( \frac{\ell_2^2}{\ell_2^2} - \frac{\ell_1^2}{\ell_2^2} \right) + E_2 \frac{\ell_1^3}{\ell_2^3}} = \frac{E_1 E_2}{E_1 (1 - \frac{\ell_1}{\ell_2}) + E_2 \frac{\ell_1}{\ell_2}}.$$

This volume element and the rest of the volume are in isostrain condition with respect to the applied stress. As a result the modulus of the composite is:

$$E_c = E_2 V_2 + E_c' V_1 = E_2 \frac{\ell_2^3}{\ell_2^3} + E_c' \frac{\ell_1^3}{\ell_2^3} = E_2 \left( 1 - \left( \frac{\ell_1}{\ell_2} \right)^2 \right) + \frac{E_1 E_2}{E_1 (1 - \frac{\ell_1}{\ell_2}) + E_2 \frac{\ell_1}{\ell_2}} \left( \frac{\ell_1^3}{\ell_2^3} \right)$$

(b) Show that this is close to lower bound:

$$E_c = \frac{E_1 E_2}{E_1 \left( \frac{\ell_2^3}{\ell_2^3} - \frac{\ell_1^3}{\ell_2^3} \right) + E_2 \frac{\ell_1^3}{\ell_2^3}}$$

by plotting both equations on a plot of $E_c$ vs. $V_1$ ($= \ell_1 / \ell_2$). In plotting, assume that $E_1 = 200 GPa$ and $E_2 = 10 GPa$.

From the graph below, the estimated modulus of particulate composite is more close to the lower bound.
3. A bilayer is made up of a 0.5 mm thick layer of aluminum on a 0.8 mm thick layer of silicon. At 320 °C, the bilayer has zero curvature. The temperature is then decreased to 20 °C. Assume that the bilayer can be treated as a beam. The properties of aluminum and silicon are as follows:

Aluminum: $E=70 \text{ GPa}$, $\nu=0.33$, $\alpha=23 \times 10^{-6} / \text{°C}$

Silicon: $E=130 \text{ GPa}$, $\nu=0.28$, $\alpha=3 \times 10^{-6} / \text{°C}$

(a) What is the curvature of the bilayer at 20 °C? Sketch the shape of the curvature.

$$\kappa = \frac{(\alpha_2 - \alpha_1)(T - T_0)}{h \left( \frac{E_1 \alpha_1^3 + E_2 \alpha_2^3}{6h} \right)} = 5.9 \text{m}^{-1}$$

(b) What is the maximum stress in the bilayer at 20 °C? At what location through the thickness of the bilayer does this stress occur? Is the stress tensile or compressive?

Maximum stress occurs at the interface.
\[ \sigma_{\text{max,Al}} = \frac{P}{a_1 b} + \frac{a_1 E_1}{2 \rho} = \frac{1}{\rho} \left[ \frac{2}{h a_1 b} (E_1 I_1 + E_2 I_2) + \frac{a_1 E_1}{2} \right] = 217 \text{ MPa (Tension)} \]

\[ \sigma_{\text{max, Si}} = -\left( \frac{P}{a_2 b} + \frac{a_2 E_2}{2 \rho} \right) = \frac{1}{\rho} \left[ \frac{2}{h a_2 b} (E_1 I_1 + E_2 I_2) + \frac{a_2 E_2}{2} \right] = -377 \text{ MPa (Compression)} \]

4. Consider a thin film of aluminum, 1\( \mu \text{m} \) in thickness, which is deposited uniformly on a Si substrate, 500\( \mu \text{m} \) in thickness and 200mm in diameter, at a temperature of 50\( ^\circ \text{C} \). The properties of the film and the substrate are the same as in Problem 3. The film-substrate system is stress-free at the deposition temperature. For simplicity, assume the film is isotropic.

(a) Find the mismatch strain and stress of the film with respect to the substrate at room temperature (20\( ^\circ \text{C} \)).

\[ (\alpha_2 - \alpha_1) \Delta T = 0.0006 \]

\[ M_{\text{film}} = \frac{E}{1 - \nu}, \text{ hence } \sigma_{\text{film}} = M_{\text{film}} \epsilon_{\text{film}} = 62.6 \text{ MPa} \]

(b) Find the curvature and the shape of the curvature.

\[ \kappa = 6 \frac{\sigma_{\text{film}}}{M_{\text{sub}}} \frac{a_1}{a_2} = 0.00832 \text{ m}^{-1} \text{ and the film is in tension so the substrate curves upward. (The exaggerated shape of curvature is the same as in Problem 3)} \]

(c) If the yield strength of the Al film is 180 MPa in both tension and compression over the temperature range 20-50\( ^\circ \text{C} \), describe the variation of curvature with temperature fluctuations over this range.

The film is stress free at the deposition temperature and attains a peak stress of 62.6MPa when cooled to 20\( ^\circ \text{C} \). Since the peak stress is below the yield strength of the film, no yielding takes place during cooling and the variation of curvature with temperature fluctuations will be linear.

5. An aluminum film with a thickness \( a_1 \) of 1\( \mu \text{m} \) is deposited on a Si substrate with a thickness \( a_2 \) of 500\( \mu \text{m} \) at an elevated temperature. The silicon substrate is a circular disk that is large compared to the thickness. After cooling to room temperature, an X-ray diffraction measurement shows that the d-spacing in the direction perpendicular to the film-substrate interface has decreased by 1%. Assume the film is isotropic and the material properties are the same as in problem 3.

(a) Calculate the strain and the stress in the film.
You are given that the strain in the direction perpendicular to the film-substrate interface \( (\varepsilon_z) \) is \(-1\%\) or \(0.01\). The film strain \( \varepsilon_{film} \) can be calculated from three dimensional Hooke's Law:

\[
\varepsilon_x = \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)], \quad \varepsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)], \quad \varepsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]
\]

For this stress state, \( \sigma_x = \sigma_y \) and \( \sigma_z = 0 \). By substituting this condition, the film strain in x-y plane is obtained as:

\[
\varepsilon_{film} = \frac{-(1-\nu)}{2\nu} \varepsilon_z = -\frac{0.67}{0.66} (-0.01) = 0.01015.
\]

Then the film stress can be determined from:

\[
\sigma_x = M_{film} \varepsilon_{film} = (104.5)(0.01015) = 1.06 \text{GPa}
\]

(b) Calculate the curvature of the wafer.

\[
\kappa = 6 \frac{\sigma_{film}}{M_{sub}} \frac{a_1}{a_2^2} = 6 \left( \frac{1.06}{180.6} \right) \frac{1 \times 10^{-6}}{(500 \times 10^{-6})^2} = 0.141 \text{m}^{-1}.
\]

(c) It is determined that the curvature is too large for the wafer to be accepted for the intended application. As a first suggestion to decrease the curvature, you recommend that the film thickness be decreased if possible. The wafer manufacturer says that they can decrease the film thickness, but for this application they must keep the ratio of the film thickness to substrate thickness, \( a_1/a_2 \) constant. Does this change your recommendation?

Consider the Stoney Formula:

\[
\kappa = 6 \frac{\sigma_{film}}{M_{sub}} \frac{a_1}{a_2^2}
\]

Thus, the curvature is proportional to film thickness. However, subject to the condition that we must keep the ratio of the film thickness to substrate thickness \( (a_1/a_2) \) constant, we can rewrite the Stoney formula as

\[
\kappa = 6 \frac{\sigma_{film}}{M_{sub}} \left( \frac{a_1^2}{a_2^2} \right) \frac{1}{a_1} = 6 \frac{\sigma_{film}}{M_{sub}} C^2 \frac{1}{a_1}
\]

where \( C=(a_1/a_2) = \) constant. Now the curvature is inversely proportional to film thickness and your recommendation must change; instead the film thickness should be increased.

6. Consider the problem of thermal stresses generated in a bimaterial strip subjected to
temperature variations. Let the strip be made by diffusion bonding between two isotropic materials whose Young’s modulus are $E_1$ and $E_2$, Poisson’s ratio are $\nu_1$ and $\nu_2$, and linear thermal expansion coefficient are $\alpha_1$ and $\alpha_2$. The yield strength of layer 1, which may be assumed to be elastic-perfectly plastic, is $\sigma_{y1}$. Layer 2 is a polycrystalline ceramic. The interface between the two layer is a perfect mechanical bond. The thickness of layer 1 and 2 are $a_1$ and $a_2$, respectively, and the other dimensions of the two layers are much larger than $a_1$ and $a_2$.

(a) Calculate the critical temperature change, $\Delta T^*$, at which layer 1 will begin to yield plastically. (Ignore edge effects)

Layer 1 will yield first at the location of maximum stress which is the interface between the two materials.

We know that for $z=0$ (at the interface),:

$$\sigma_{\text{max}} = \sigma_1(z = 0) = \frac{P}{a_1b} + \frac{a_1}{2\rho} = \frac{1}{\rho} \left[ \frac{2}{ha_1b} (E_1I_1 + E_2I_2) + \frac{a_1E_1}{2} \right]$$

where $\kappa = \frac{1}{\rho} = \frac{(\alpha_2 - \alpha_1)(T - T_0)}{\frac{h}{2} \left[ \frac{1}{E_1a_1b} + \frac{1}{E_2a_2b} \right]}$.

For yielding to occur, $\sigma_{\text{max}} = \sigma_{y1}$ when $\Delta T = \Delta T^*$.

As a result, we can solve for $\Delta T^*$ as

$$\Delta T^* = \frac{\sigma_{y1} \left[ \frac{1}{2} + \left( \frac{2(E_1I_1 + E_2I_2)}{h} \left( \frac{1}{a_1E_1b} + \frac{1}{a_2E_2b} \right) \right) \right]}{(\alpha_2 - \alpha_1) \left[ \frac{2(E_1I_1 + E_2I_2)}{ha_1b} + \frac{a_1E_1}{2} \right]}$$

(b) Indicate the location at which yielding commences. Describe the spread of plastic zone with increasing $\Delta T$.

Yielding commences at the interface and spreads towards the outer surface.

(c) What is the curvature of the strip for a temperature change of $\Delta T^*$?

When $\Delta T = \Delta T^*$,

$$\kappa = \frac{1}{\rho} = \frac{(\alpha_2 - \alpha_1)\Delta T^*}{\frac{h}{2} \left[ \frac{1}{E_1a_1b} + \frac{1}{E_2a_2b} \right]}$$

(d) If the strip were to be subjected to a uniform bending moment $M$ instead of a temperature change, what is the location at which yielding commences and how does the plastic zone spread with increasing $M$?
If the strip is subjected to a bending moment $M$, the highest tensile or compressive stress will be at the outer surface and hence plastic yielding will commence there. It will then spread inward toward the interface.

(e) Schematically sketch the variation of temperature versus curvature for both increasing and decreasing $\Delta T$ for the following two cases: (i) $\Delta T < \Delta T^*$, and (ii) $\Delta T > \Delta T^*$, where all $\Delta$ quantities are absolute values.

(i) If there is no yielding, curvature varies linearly with $\Delta T$ and retraces itself.

(ii) If yielding occurs, there is a change in slope and the curve does not retrace itself due to the permanent deformation of yielding.