1. A thin-walled pressure vessel is subjected to pressure cycles in which the internal pressure ranges from 2 to 10 MPa. The inner diameter and thickness of the vessel are 150 cm and 5 cm, respectively. The vessel is made from an aluminum alloy with $\sigma_{ys} = 350$ MPa, and $K_{fc} = 25 \text{ MPa}\sqrt{\text{m}}$. We wish to consider the failure of the vessel caused by growth of fatigue cracks. Consider the growth of semicircular cracks inside the vessel, oriented normal to the hoop stress. The stress intensity factor for such cracks is:

$$K_I = \frac{1.12\sqrt{\pi a}}{Q} \approx \frac{2.24}{\pi} \frac{\sigma}{\sqrt{\pi a}}$$

Assume that the growth of fatigue cracks in the plate is governed by a Paris type of law, i.e.

$$\frac{da}{dN} = C(\Delta K)^2$$

where $C = 3.75 \times 10^{-8} (\text{MPa}\sqrt{\text{m}})^{-2}\cdot\text{m}$

For the present case, the hoop stress is related to the radius of the pressure vessel $r$, its wall thickness $t$, and the pressure $P$ as

$$\sigma_{hoop} = \frac{Pr}{t}$$

(a) It is desired that the pressure vessel should survive for at least 4,000 pressure cycles. In order for the vessel to survive for the desired number of cycles, the initial size of any potential fatigue crack must be less than a certain size. Calculate that crack size.

(15 points)
(b) One method to ensure that the vessel will survive for the desired number of cycles is to carry out some type of non-destructive crack inspection technique (x-ray, ultrasonics, etc.) and verify that there are no cracks at or above the size you calculated in (a). Another method is to pressurize the vessel (under controlled, safe conditions) to a test pressure at which the vessel would fail by fracture if a crack of that size were present. If the vessel survives the test pressure, one may conclude that no cracks of that size are present. Calculate the test pressure required. Can the vessel sustain this pressure without yielding?

(10 points)

2. A cracked plate of a material with Young’s modulus $E = 110$ GPa, Poisson ratio $\nu = 0.33$, and yield strength $\sigma_{YS} = 400$ MPa, is subjected to a mode I stress intensity factor, $K_1 = 10$ MPa$\sqrt{m}$.

a) Determine the expressions for the radius of the mode I plastic zone (for plane stress and plane strain) using the Tresca yield criterion.

b) Evaluate the ratio of the plane strain radius to the plane stress radius for $\theta = 0^\circ$ and $\theta = 45^\circ$.

c) Calculate the extent of the plane strain plastic zone at $\theta = 0^\circ$.

Recall that the Tresca yield criterion states that yielding occurs when the maximum shear stress reaches a critical value, i.e.

$$\tau_{\text{max}} = \frac{1}{2}\left|\sigma_{\text{max}} - \sigma_{\text{min}}\right| = \frac{\sigma_{YS}}{2}$$

Note that the principal stresses $\sigma_1, \sigma_2$ for mode I loading are given by

$$\sigma_1 = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2})$$

$$\sigma_2 = \frac{K_1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2})$$

(40 points)
3. (a) Consider a tension test on sample of silicon nitride with $K_{ic} = 6 \text{ MPa} \sqrt{\text{m}}$. Typically, the tensile strength $\sigma_t$ of this material is $\approx 500 \text{ MPa}$. Assuming that failure in tension is initiated from surface flaws (or cracks), use this value for the tensile strength to estimate the size (radius $a$) of a typical surface flaw that causes failure of the sample. Assume that the surface flaws are in the shape of semi-circular cracks, for which the stress intensity factor is given as:

$$K_I = \frac{1.12 \sigma \sqrt{\pi a}}{\sqrt{Q}} \approx \frac{2.24}{\pi} \sigma \sqrt{\pi a}$$

(10 points)

(b) For brittle materials such as silicon nitride, since they do not undergo significant plasticity there may be no (dislocation) plastic zone at the crack tip. There is, however, a process zone, where microcracking or some other type of damage process occurs, and the $K$ solution is not valid in this region. The dimensions of this process zone can be found simply by substituting the tensile strength in place of the yield strength $\sigma_{YS}$ in the expressions derived in class for the plastic zone size. For this silicon nitride determine the approximate size of the process zone (for both plane stress and plane strain) using the approximate “engineering” expressions. Then determine the approximate dimensions (crack length $a$, width $W$ and thickness $B$) required to perform a valid plane strain $K_{ic}$ test for this material using a compact tension specimen.

(10 points)

(c) A plate of a coarse grained ceramic material, with grain size = 30 $\mu$m, contains a 0.1 mm-long edge crack. The mode I stress intensity factor to which the crack is subjected is $1 \text{ MPa} \sqrt{\text{m}}$. The tensile strength of the ceramic is 300 MPa. Comment on the validity of the use of the stress intensity factor to characterize fracture in this material.

(15 points)
1. (a) We are given a thin-walled pressure vessel, which is subjected to pressure cycles in which the internal pressure ranges from 2 to 10 MPa. The inner diameter and thickness of the vessel are 150 cm and 5 cm, respectively. We assume we have cracks oriented normal to the hoop stress. For cracks of this orientation the stresses $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ (hoop stress) are given as

$$\sigma_{\text{max}} = \frac{P_{\text{max}}r}{t} = 15P_{\text{max}} = 150 \text{ MPa}$$

$$\sigma_{\text{min}} = \frac{P_{\text{min}}r}{t} = 15P_{\text{min}} = 30 \text{ MPa}$$

We first calculate the critical crack length by setting the given stress intensity factor (with $\sigma = \sigma_{\text{max}}$) equal to $K_{\text{fc}}$ and find that:

$$a_f = \frac{1}{\pi} \left( \frac{\pi}{2.24} \right)^2 \left( \frac{K_{\text{fc}}}{\sigma_{\text{max}}} \right)^2 = 1.74 \text{ cm}$$

This is less than the thickness, so the vessel will not leak before break. If we assume that the growth of fatigue cracks in the plate is governed by a Paris type of law with $m = 2$, the lifetime of the plate is given by (either integrate or use the result given as Eqn. 10.9 in the text)

$$N_f = \frac{\ln \left( \frac{a_f}{a_0} \right)}{CY^2 (\Delta \sigma)^2 \pi}$$

In this case $N_f$ is required to be 4,000, so everything is known except $a_0$. We may solve for $a_0$ to find that

$$a_0 = 0.553 \text{ mm}$$
This is a fairly small crack but would likely be detected by most inspection techniques.

(b) Now assume a crack of size \( a = 0.553 \) mm exists. Set \( K_I = K_{ic} \) in the given expression for the stress intensity factor and solve for \( \sigma \), then determine the test pressure required to cause that stress. Find that

\[
\sigma = \frac{\pi}{2.24} \frac{K_{ic}}{\sqrt{\pi a}} = 841 \text{ MPa}
\]

From before, we know that \( \sigma = Pr/t = 15P \), so that \( P_{test} = \sigma/15 = 56 \text{ MPa} \). Since this test stress is above the yield stress of the material, the vessel would yield before failing by fracture and thus this is not a viable method for determining the presence of the crack under these circumstances.

(Use either Von Mises or Tresca to determine if the vessel would yield.)
The Tresca criterion states that yield occurs when the maximum shear stress reaches a critical value, i.e.

\[ \tau_{\text{max}} = \frac{1}{2} |\sigma_{\text{max}} - \sigma_{\text{min}}| = \frac{\sigma_{\text{YS}}}{2} \]

So the only trick in the problem is to find out what are the maximum and minimum stresses. Looking at the form of the principal stresses (see page 7 in the notes on plastic zone size), it can be seen that for \( \theta \) between -180 and 180 degrees, \( \sigma_1 \) is always \( \sigma_{\text{max}} \). What about \( \sigma_{\text{min}} \)? For the plane stress case, \( \sigma_3 = 0 \) (and \( \sigma_2 \) is never negative) and thus \( \sigma_3 = \sigma_{\text{min}} \) so that

\[ \frac{1}{2} |\sigma_{\text{max}} - \sigma_{\text{min}}| = \frac{1}{2} |\sigma_1 - 0| = \frac{K_I}{\sqrt{2\pi r_p}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2}\right) = \frac{\sigma_{\text{YS}}}{2} \]

solve for \( r_p \)

\[ \frac{r_p}{r_p^*} = \cos^2 \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2}\right)^2 \]

where \( r_p^* = K_I^2/(2\pi \sigma_{\text{YS}}^2) \) is the approximate plastic zone size (see page 2 on the notes on the plastic zone size). For the plane strain case, we need to decide which stress (\( \sigma_2 \) or \( \sigma_3 \)) is the minimum stress. First evaluate the stresses \( \sigma_2 \) and \( \sigma_3 \) at \( \theta = 0^\circ \). We find that

\[ \sigma_2 = \frac{K_I}{2\pi r^2}, \quad \sigma_3 = 2\nu \frac{K_I}{2\pi r} \]

For a reasonable value of \( \nu \) (say \( \nu = 1/3 \)) we see that \( \sigma_3 < \sigma_2 \) so \( \sigma_3 = \sigma_{\text{min}} \). But note that at some critical angle \( \theta_c \), \( \sigma_3 \) is equal to \( \sigma_2 \),
and beyond that angle $\sigma_2$ is now $\sigma_{\text{min}}$. So the minimum stress $\sigma_{\text{min}}$ is either $\sigma_2$ or $\sigma_3$, depending on the angle. We can find the critical angle by setting $\sigma_3 = \sigma_2$:

$$\frac{K_I}{2\pi r} \cos \frac{\theta_c}{2} \left(1 - \sin \frac{\theta_c}{2}\right) = \frac{K_I}{2\pi r} 2\nu \cos \frac{\theta_c}{2}$$

or

$$\left(1 - \sin \frac{\theta_c}{2}\right) = 2\nu$$

Solve for $\theta_c$:

$$\theta_c = 2\sin^{-1}(1 - 2\nu)$$

For $\theta < \theta_c$, $\sigma_3 = \sigma_{\text{min}}$ and the plastic zone radius is given by:

$$\frac{r_p}{r_p^*} = \cos^2 \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} - 2\nu\right)^2$$

and for $\theta > \theta_c$, $\sigma_2 = \sigma_{\text{min}}$ and the plastic zone radius is given by:

$$\frac{r_p}{r_p^*} = 4\cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2}$$

In the problem we are also asked to consider the ratio of the plastic zone radius for plane strain and plane stress for $\theta = 0^\circ$ and $\theta = 45^\circ$. For the calculation, I assumed that $\nu$ was equal to 1/3 and thus $\theta_c = \cdot$. The plastic zone radii are evaluated below:

$$\frac{r_p}{r_p^*} = 1 \text{ (Plane Stress, } \theta = 0^\circ)$$

$$\frac{r_p}{r_p^*} = 1.63 \text{ (Plane Stress, } \theta = 45^\circ)$$

$$\frac{r_p}{r_p^*} = \frac{1}{9} = 0.111 \text{ (Plane Strain, } \theta = 0^\circ)$$
\[
\frac{r_p}{r_{ps}} = 0.5 \text{ (Plane Strain, } \theta = 45^\circ) \\
\]

and thus the ratios are given as:

\[
\frac{r_p \text{ (Plane Strain)}}{r_p \text{ (Plane Stress)}} = \frac{1}{9} = 0.111 \text{ (} \theta = 0^\circ \text{)}
\]

\[
\frac{r_p \text{ (Plane Strain)}}{r_p \text{ (Plane Stress)}} = \frac{0.5}{1.63} = 0.307 \text{ (} \theta = 45^\circ \text{)}
\]

3. (a) Simply set the stress intensity factor \( K_I \) equal to \( K_{ic} \), set the stress \( \sigma = \sigma_t \), and solve for \( a \):

\[
a = \frac{1}{\pi} \left( \frac{K_{ic}}{\sigma} \right)^2 \left( \frac{\pi}{2.24} \right)^2 = 90.2 \ \mu m
\]

This is a very small flaw and would likely be difficult to detect.

(b) Use the approximations for the size of the inelastic zone (replace \( \sigma_{YS} \) by \( \sigma_t \)):

\[
r_p = \frac{1}{\pi} \left( \frac{K_{ic}}{\sigma_t} \right)^2 = 15.3 \ \mu m \text{ (plane stress)}
\]

\[
r_p = \frac{1}{3\pi} \left( \frac{K_{ic}}{\sigma_t} \right)^2 = 45.8 \ \mu m \text{ (plane strain)}
\]

For a valid \( K_{ic} \) test all the specimen dimensions \((B, a, W - a)\) should be at least 25 times the plane strain inelastic zone. So we may set \( a = W = B = 382 \ \mu m \). This is a very small sample! In fact, too small to test on conventional mechanical testing equipment. Typically, for brittle materials it is very easy to satisfy the specimen size conditions for a valid \( K_{ic} \) test with a reasonably sized sample.

(c)

\[
r_0 \approx 0.1a = 0.01mm = 10\mu m; r_{\text{grain}} = 30\mu m; r_p \approx \frac{1}{3\pi} \left( \frac{1}{300} \right)^2 \approx 1\mu m
\]

So \( r_p \) is less than \( r_{\text{grain}} \). In such a single grain, crystallography dominates, and the use of the stress intensity factor, which is an isotropic continuum concept, will not work well.