**Example Problem**

BTIV: Review Sheets 05.06.04

Fernandez Intro.

As in the example problems in class, this derivation of a material index is based on the recognition that design situations usually involve significant knowledge about function and geometry (configuration). Therefore the final (simple) equation should be expressed only in terms of material properties. This final expression can then be used as a "design guideline" in a multi-objective optimization search. (See CES4.1, www.grantadesign.com).

I will ask, in the single exam question, that you derive a material index. Please follow the steps below.

1. **Complete a Design Requirement Table**

<table>
<thead>
<tr>
<th>Function</th>
<th>What is Essential Function?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Objective</td>
<td>What is being optimized (min. or max)?</td>
</tr>
<tr>
<td>Constraints</td>
<td>What are parameters (geometric)?</td>
</tr>
</tbody>
</table>

2. **Formulate Equations**

Usually equations will come from both objective and constraint statements. For example:

\[ S = \frac{F}{8} \] (stiffness equation)

\[ m = \alpha \rho \] (mass equation)

Eq. 3 \[ F_t = \frac{C_2 I_0 t}{Y m} \] (failure load of a beam)
Note: Equation 3 on previous page is type of equation that I will give you. This type includes constitutive equations for mechanical response.

Other examples:

\[ F_{cr,1} = \frac{n^2 \pi^2 EI}{k^2} \quad \text{Column} \]
\[ \delta = \frac{Fl^3}{C_1EI} = \frac{Ml^2}{C_1EI} \quad \text{Beam (Deflection, } \delta \text{)} \]

3. Combine and simplify equations in structural situations, usually the best path toward a solution involves eliminating a (area) by combining and simplifying equations.

Example from class:

\[ m = A \rho \quad \Rightarrow \quad \frac{\sigma_f \geq \frac{F}{A}}{A = \frac{m}{\rho}} \]

\[ \sigma_f \geq \frac{Fl\rho}{m} \]

4. Organize final equation only in terms of material properties.

Again, using example above:

\[ \sigma_f \geq \frac{Fl\rho}{m} \Rightarrow m \geq (F \times l) \left( \frac{\rho}{\sigma_f} \right) \]
Therefore, we can now apply our assumption
that we know about the load and the
geometry and the material index ($M_i$)
can be expressed as a simple ratio:

$$M_i \approx \frac{P}{\sigma_f}$$

Note: $M_i$ is not mass. It is an index
of performance. We are trying
to minimize $M_i$ and therefore
we can also write:

$$M_i \approx \frac{\sigma_f}{P}$$

As a material index, we are trying to maximize $M_i$.

Now, let's look at an index
that minimizes cost (an extrinsic
property).
Material Index for a Minimal Cost, Stiff Column.

1. Design Requirements
   Function: Column (Axially Loaded)
   Objective: Minimize Cost
   Constraints: a) Length, \( l \)
   b) Critical (Buckling) Load, \( F_{\text{crit}} \)
   Note: We are assuming this is a 'Long' Column, and will therefore fail in Buckling, not Crushing

2. Formulate Equations
   \[
   \text{Eq. 1} \quad C = A l C_m p
   \]
   This is the Objective, and we are trying to Min. C.

   \[
   \text{Eq. 2} \quad F \leq F_{\text{crit}} = \frac{w T^2 E I}{l^2}
   \]
   This equation combines constraints variables with a mathematical expression of the function

* Before you move on, note that the variables that are purely based on the material's properties are: \( C, C_m, \rho, E \)
All else is either known and/or not based on material properties.
Also note that I will give you EQ. 2. I would not "give" you EQ. 1, but I would state that cost should be expressed in terms of price per unit mass.

Therefore: \[ m = A \rho, \quad C_m = \frac{C}{m} \]

And, \[ C_m = \frac{C}{A \rho} \Rightarrow C = A \rho C_m \quad \text{(EQ. 1)} \]

Ok, now Step Three:

3. Combine and Simplify

I will give you the following

\[ \text{EQ. 3 } I = \frac{\pi r^2}{4} = \frac{A^2}{4\pi} \]

Insert EQ. 3 into EQ. 2

\[ \text{EQ. 2 } F_{\text{crt}} = \frac{n \pi \rho^2}{l^2} (I) \Rightarrow F_{\text{crt}} = \frac{n \pi \rho^2}{l^2} \left( \frac{A^2}{4\pi} \right) \]

Now use EQ. 1, in terms of \( A \),

\[ \text{EQ. 1 } C = A l C_m \rho \Rightarrow A = \frac{C}{l C_m \rho} \]

And insert \( A \) into EQ. 2

\[ F = \frac{n \pi \rho^2}{l^2} \left( \frac{A^2}{4\pi} \right) \]

And, you get the following

\[ F_{\text{crt}} = \frac{n \pi \rho^2}{l^2} \left( \frac{C^2}{l^2 C_m^2 \rho^2 4\pi} \right) \]

We want to minimize \( C \), so isolate \( C \) on one side of equation...
\[ F_{\text{crit}} = \left( \frac{n \pi^2 E}{l^4 C_m^2 \rho^2 4 \pi^2} \right) (C^2) \]

AND,

\[ \text{EQ.4} \quad C^2 = F_{\text{crit}} \left( \frac{l^4 C_m^2 \rho^2}{\pi^2} \right) \left( \frac{4 \pi}{n E} \right) \]

Now, it's time to enter Step Four.

4. Organize final equation only in terms of material properties.

What are terms that remain that are purely dependent (intrinsic or extrinsic) on material properties?

(Note, I've taken the square root of both sides of EQ.4)

\[ C = (F_{\text{crit}})^{1/2} \left( \frac{l^2 C_m \rho}{\pi} \right) \left( \frac{4 \pi}{n E} \right)^{1/2} \]

\[ \uparrow \quad \text{Load} \quad \uparrow \quad \text{Geo.} \quad \uparrow \quad \text{Constant} \]

\[ \downarrow \quad \text{Constant} \]

What's left? \( C, C_m, \rho \) and \( E \)

\[ \rightarrow m_i = \frac{C_m \rho}{E^{1/2}} \]

Here, \( m_i \) is minimized.

\[ \begin{vmatrix} \text{OR} \end{vmatrix} \]

\[ m_i = \frac{E^{1/2}}{C_m \rho} \]

Here, \( m_i \) is maximized.

C'EST TOUT.