Pictures help to accurately approximate $\ln n!$

$$3! = 3 \times 2 \times 1;$$
$$4! = 4 \times 3 \times 2 \times 1;$$
$$5! = 5 \times 4 \times 3 \times 2 \times 1;$$
$$\ldots$$

$n!$ is the most important function in statistical mechanics.
\[ \ln n! \text{ is the area of the rectangles} \]

\[ \ln n! = \sum_{1}^{n} \ln k \]
The area under $\ln k$ is the first approximation

$$\ln n! \approx \int_1^n \ln k \, dk = n \ln n - n + 1$$
The error is the protruding pieces
Each piece is almost a triangle
Doubling the ‘triangles’ makes them easier to add
The rectangles slide across and stack at the end.

Sum of doubled protrusions = $\ln n$
Combine the integral and approximated protrusion

\[ \ln n! = \sum_{1}^{n} \ln k \]

\[ \approx n \ln n - n + 1 + \frac{1}{2} \ln n \]
The preceding pictorial approximation ignores only a tiny region
Numerical calculation confirms the accuracy

Picture: \[ 7 \times (\ln 7 - 1) + 1 + \frac{1}{2} \ln 7 = 8.594 \ldots \ldots \]

Exact: \[ \sum_{1}^{7} \ln k = 8.525 \ldots \ldots \]

The approximation makes error of 0.07 in \( \ln 7! \) (which results in a 7% error in 7!).