cube has six faces, so six pyramids might make a cube where each pyramid base forms one face of the cube, and each pyramid tip faces inward, meeting in the center of the cube. For the tips to meet in the center of the cube, the height must be \( h = 1 \). So six pyramids with \( b = 2 \), and \( h = 1 \) make a cube with side length 2.

The volume of one pyramid is one-sixth of the volume of the cube:

\[
V = \frac{\text{cube volume}}{6} = \frac{8}{6} = \frac{4}{3}.
\]

The volume of the pyramid is \( V \sim hb^2 \), and the missing constant must make volume \( 4/3 \). Since \( hb^2 = 4 \) for these pyramids, the missing constant is \( 1/3 \). Voilà:

\[
V = \frac{1}{3} hb^2 = \frac{4}{3}.
\]

### 7.2 Mechanics

#### 7.2.1 Atwood machine

The next problem illustrates dimensional analysis and special cases in a physical problem. Many of the ideas and methods from the geometry example transfer to this problem, and it introduces more methods and ways of reasoning.

The problem is a staple of first-year physics: Two masses, \( m_1 \) and \( m_2 \), are connected and, thanks to a pulley, are free to move up and down. What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.

The first problem is to find the acceleration of, say, \( m_1 \). Since \( m_1 \) and \( m_2 \) are connected by a rope, the acceleration of \( m_2 \) is, depending on your sign convention, either equal to \( m_1 \) or equal to \(-m_1 \). Let’s call the acceleration \( a \) and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so \( g \) should be on the list. The masses affect the acceleration, so \( m_1 \) and \( m_2 \) are on the list. And that’s it. You might wonder what happened to the tension: Doesn’t it affect the acceleration? It does, but it is itself a consequence of \( m_1 \), \( m_2 \), and \( g \). So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.
These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: \( G_1 = \frac{m_1}{m_2} \) and \( G_2 = \frac{a}{g} \). You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

\[
\frac{a}{g} = f \left( \frac{m_1}{m_2} \right),
\]

where \( f \) is a dimensionless function.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find \( f \) using \( \frac{m_1}{m_2} \) as the dimensionless group, first choose a better group. The ratio \( \frac{m_1}{m_2} \) does not respect the symmetry of the problem in that only the sign of the acceleration changes when you interchange the labels \( m_1 \) and \( m_2 \). Whereas \( \frac{m_1}{m_2} \) turns into its reciprocal. So the function \( f \) will have to do lots of work to turn the unsymmetric ratio \( \frac{m_1}{m_2} \) into a symmetric acceleration.

Back to the drawing board for how to fix \( G_1 \). Another option is to use \( m_1 - m_2 \). Wait, the difference is not dimensionless! I fix that problem in a moment. For now observe the virtue of \( m_1 - m_2 \). It shows a physically reasonable symmetry under mass interchange: \( G_1 \rightarrow -G_1 \). To make it dimensionless, divide it by another mass. One candidate is \( m_1 \):

\[
G_1 = \frac{m_1 - m_2}{m_1},
\]

That choice, like dividing by \( m_2 \), abandons the beloved symmetry. But dividing by \( m_1 + m_2 \) solves all the problems:

\[
G_1 = \frac{m_1 - m_2}{m_1 + m_2}.
\]

This group is dimensionless and it respects the symmetry of the problem.

Using this \( G_1 \), the solution becomes

\[
\frac{a}{g} = f \left( \frac{m_1 - m_2}{m_1 + m_2} \right),
\]

where \( f \) is another dimensionless function.

<table>
<thead>
<tr>
<th>Var</th>
<th>Dim</th>
<th>What</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>LT(^{-2})</td>
<td>accel. of ( m_1 )</td>
</tr>
<tr>
<td>( g )</td>
<td>LT(^{-2})</td>
<td>gravity</td>
</tr>
<tr>
<td>( m_1 )</td>
<td>M</td>
<td>block mass</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>M</td>
<td>block mass</td>
</tr>
</tbody>
</table>
To guess \( f(x) \), where \( x = G_1 \), try special cases. First imagine that \( m_1 \) becomes huge. A quantity with mass cannot be huge on its own, however. Here huge means huge relative to \( m_2 \), whereupon \( x \approx 1 \). In this thought experiment, \( m_1 \) falls as if there were no \( m_2 \) so \( a = -g \). Here we’ve chosen a sign convention with positive acceleration being upward. If \( m_2 \) is huge relative to \( m_1 \), which means \( x = -1 \), then \( m_2 \) falls like a stone pulling \( m_1 \) upward with acceleration \( a = g \). A third limiting case is \( m_1 = m_2 \) or \( x = 0 \), whereupon the masses are in equilibrium so \( a = 0 \).

Here is a plot of our knowledge of \( f \):

\[
\begin{array}{c}
\text{1} \\
\text{} \\
\text{} \\
\text{-1} \\
\end{array}
\]

\[
\begin{array}{c}
\text{1} \\
\text{f(x)} \\
\text{} \\
\text{-1} \\
\end{array}
\]

\[
\begin{array}{c}
\text{-1} \\
\text{x} \\
\text{1} \\
\end{array}
\]

The simplest conjecture – an educated guess – is that \( f(x) = x \). Then we have our result:

\[
\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.
\]

Look how simple the result is when derived in a symmetric, dimensionless form using special cases!

### 7.3 Drag

Pendulum motion is not a horrible enough problem to show the full benefit of dimensional analysis. Instead try fluid mechanics – a subject notorious for its mathematical and physical complexity; Chandrasekhar’s books [6, 7] or the classic textbook of Lamb [19] show that the mathematics is not for the faint of heart.