Massachusetts Institute of Technology
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6.061 Supplementary Notes 1
Review of Network Theory

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1 Introduction

This note is a review of some of the most salient points of electric network theory. In it we do not prove any of the assertions that are made. We will deal only with passive, linear network elements.

2 Network Primitives

Electric network theory deals with two primitive quantities, which we will refer to as:

1. Potential (or voltage), and
2. Current.

Current is the actual flow of charged carriers, while difference in potential is the force that causes that flow. As we will see, potential is a single-valued function that may be uniquely defined over the nodes of a network. Current, on the other hand, flows through the branches of the network. Figure 1 shows the basic notion of a branch, in which a voltage is defined across the branch and a current is defined to flow through the branch. A network is a collection of such elements, connected together by wires.

\[
\begin{array}{c}
\text{i} + \text{v.} \\
\hline
\end{array}
\]

Figure 1: Basic Circuit Element

Network topology is determined by the interconnection of its elements. That, plus the constraints on voltage and current imposed by the elements themselves, determines the performance of the network, described by the distribution of voltages and currents throughout the network.

Two important concepts must be described initially. These are of “loop” and “node”.

1
1. A *loop* in the network is any *closed* path through two or more elements of the network. Any non-trivial network will have at least one such loop.

2. A *node* is a point at which two or more elements are interconnected.

The two fundamental laws of network theory are known as *Kirchoff’s Voltage Law* (KVL), and *Kirchoff’s Current Law* (KCL). These laws describe the topology of the network, and arise directly from the fundamental laws of electromagnetics. They are simply stated as:

- Kirchoff’s Voltage Law states that, around any *loop* of a network, the sum of all voltages, taken in the same direction, is zero:
  \[ \sum_{\text{loop}} v_k = 0 \]  
  \hspace{1cm} (1) 

- Kirchoff’s Current Law states that, at any *node* of a network, the sum of all currents entering the node is zero:
  \[ \sum_{\text{node}} i_k = 0 \]  
  \hspace{1cm} (2)

The elementary network elements affect voltages and currents in one of three ways:

1. *Voltage* sources constrain the potential difference across their terminals to be of some fixed value (the value of the source).

2. *Current* sources constrain the current through the branch to be of some fixed value.
3. All other elements impose some sort of relationship, either linear or nonlinear, between voltage across and current through the branch.

Voltage and current sources can be either independent or dependent. Independent sources have values which are, as the name implies, independent of other variables in a circuit. Dependent sources have values which depend on some other variable in a circuit. A common example of a dependent source is the equivalent current source used for modeling the collector junction in a transistor. Typically, this is modeled as a current dependent current source, in which collector current is taken to be directly dependent on emitter current. Such dependent sources must be handled with some care, for certain tricks we will be discussing below do not work with them. For the most part, we will not be concerned with dependent sources here.

For the present time, we will consider, in addition to the two types of sources, only impedance elements, which impose a linear relationship between voltage and current. The most common of these is the resistance, which imposes the relationship which is often referred to as Ohm's law:

\[ v_r = R i_r \]  \hspace{1cm} (3)

\[
\begin{array}{c}
\text{Figure 5: Resistance Circuit Element}
\end{array}
\]

A bit later on in this note, we will extend this notion of impedance to other elements, but for the moment the resistance will serve our purposes.

3 Examples: Voltage and Current Dividers

Figure 6 may be used as an example to show how we use all of this. See that it has one loop and three nodes. Around the loop, KVL is:

\[ V_s - v_1 - v_2 = 0 \]
At the upper right-hand node, we have, by KCL:

\[ i_1 - i_2 = 0 \]

The constitutive relations imposed by the resistances are:

\[ v_1 = R_1 i_1 \]
\[ v_2 = R_2 i_2 \]

Combining these, we find that:

\[ V_s = (R_1 + R_2)i_1 \]

We may solve for the voltage across, say, \( R_2 \), to obtain the so-called voltage divider relationship:

\[ v_2 = V_s \frac{R_2}{R_1 + R_2} \]  \hspace{1cm} (4)

![Diagram of Voltage Divider](image)

Figure 6: Voltage Divider

A second example is illustrated by Figure 7. Here, KCL at the top node yields:

\[ I_s - i_1 - i_2 = 0 \]

And KVL, written around the loop that has the two resistances, is:

\[ R_1 i_1 - R_2 i_2 = 0 \]

Combining these together, we have the current divider relationship:

\[ i_2 = I_s \frac{R_1}{R_1 + R_2} \]  \hspace{1cm} (5)

Once we have derived the voltage and current divider relationships, we can use them as part of our “intellectual toolkit”, because they will always be true.

4 Node Voltages and Reference

One of the consequences of KVL is that every node in a network will have a potential which is uniquely specified with respect to some other node. Thus, if we take one of the nodes in the network to be a reference, or datum, each of the other nodes will have a unique potential. The voltage across any network branch is then the difference between the potentials at the nodes to which the element
is connected. The potential of a node is the sum of voltages encountered when traversing some path between that node and the datum node. Note that any path will do. If KVL is satisfied, all paths between each pair of nodes will yield the same potential.

A commonly used electric circuit is the Wheatstone Bridge, shown in its simplest form in Figure 8. The output voltage is found simply from the input voltage as just the difference between two voltage dividers:

$$v_o = v_s \left( \frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4} \right)$$

Figure 8: Elementary Wheatstone Bridge

5 Serial and Parallel Combinations

There are a number of techniques for handling network problems, and we will not be able to investigate each of them in depth. We will, however, look into a few techniques for analysis which involve progressive simplification of the network. To start, we consider how one might handle series and parallel combinations of elements. A pair of elements is in series if the same current flows through both of them. If these elements are resistors and if the detail of voltage division between them is not required, it is possible to lump the two together as a single resistance. This is illustrated in Figure 9. The voltage across the current source is:

$$v_s = v_1 + v_2 = i_s R_1 + i_s R_2 = i_s (R_1 + R_2)$$

The equivalent resistance for the series combination is then:
Figure 9: Series Resistance Combination

\[ R_{series} = R_1 + R_2 \]  \hspace{1cm} (6)

Similarly, resistance elements connected in parallel can be lumped if it is not necessary to know the details of division of current between them. Figure 10 shows this combination. Here, current

\[ i = \frac{v}{R_1} + \frac{v}{R_2} = v \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]

The equivalent resistance for the parallel combination is then:

\[ R_{par} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_1 R_2}{R_1 + R_2} \]  \hspace{1cm} (7)

Because of the importance of parallel connection of resistances (and of other impedances), a special symbolic form is used for parallel construction. This is:

\[ R_1 || R_2 = \frac{R_1 R_2}{R_1 + R_2} \]  \hspace{1cm} (8)

As an example, consider the circuit shown in Figure 11. Here, we have four, resistors arranged in an odd way to form a two-terminal network. To find the equivalent resistance of this thing, we can do a series of series-parallel combinations.

The two resistors on the right can be combined as a series combination to form a single, two ohm resistor as shown in Figure 12. Then the equivalent resistor, which is in parallel with one of the two ohm resistors can be combined to form a single combination. That is in series with the remaining resistor, leaving us with an equivalent input resistance of \( R = 3 \Omega \).
6 Loop and Node Equations

There are two well-developed formal ways of solving for the potentials and currents in networks, often referred to as loop and node equation methods. They are closely related, using KCL and KVL together with element constraints to build sets of equations which may then be solved for potentials and currents.

- In the node equation method, KCL is written at each node of the network, with currents expressed in terms of the node potentials. KVL is satisfied because the node potentials are unique.

- In the loop equation method, KVL is written about a collection of closed paths in the network. “Loop currents” are defined, and made to satisfy KCL, and the branch voltages are expressed in terms of them.

The two methods are equivalent and a choice between them is usually a matter of personal preference. The node equation method is probably more widely used, and lends itself well to computer analysis.

To illustrate how these methods work, consider the network of Figure 14. This network has three nodes. We are going to write KCL for each of the nodes, but note that only two explicit equations are required. If KCL is satisfied at two of the nodes, it is automatically
Figure 14: Sample Network

satisfied at the third. Usually the datum node is the one for which we do not write the expression. KCL written for the two upper nodes of the network is:

\[ -i_1 + \frac{V}{R_1} + \frac{V}{R_2} = 0 \]  \hfill (9)

\[ -I + \frac{v_2 - V}{R_2} + \frac{v_2}{R_3} = 0 \]  \hfill (10)

These two expressions are easily solved for the two unknowns, \( i_1 \) and \( v_2 \):

\[ v_2 = \frac{R_3}{R_2 + R_3}V + \frac{R_2R_3}{R_2 + R_3}I \]

\[ i_1 = \frac{R_1 + R_2 + R_3}{R_1(R_2 + R_3)}V - \frac{R_3}{R_2 + R_3}I \]

The loop equation method is similar. We need the same number of independent expressions (two), so we need to take two independent loops. For this, take as the loops:

1. The loop that includes the voltage source and \( R_1 \).

2. The loop that inculdes \( R_1, R_2, \) and \( R_3 \).

It is also necessary to define loop currents, which we will denote as \( i_a \) and \( i_b \). These are the currents circulating around the two loops. Note that where the loops intersect, the actual branch current will be the sum of or difference between loop currents. For this example, assume the loop currents are defined to be circulating counter-clockwise in the two loops. The two loop equations are:

\[ V + R_1(i_a - i_b) = 0 \]  \hfill (11)

\[ R_1(i_b - i_a) + R_2i_b + R_3(i_b - I) = 0 \]  \hfill (12)

These are equally easily solved for the two unknowns, in this case the two loop currents \( i_a \) and \( i_b \).

7 Linearity and Superposition

An extraordinarily powerful notion of network theory is linearity. This property has two essential elements, stated as follows:

1. For any single input \( x \) yielding output \( y \), the response to an input \( kx \) is \( ky \) for any value of \( k \).

2. If, in a multi-input network the input \( x_1 \) by itself yields output \( y_1 \) and a second input \( x_2 \) by itself yields \( y_2 \), then the combination of inputs \( x_1 \) and \( x_2 \) yields the output \( y = y_1 + y_2 \).
This is important to us at the present moment for two reasons:

1. It tells us that the solution to certain problems involving networks with multiple inputs is actually easier than we might expect: if a network is linear, we may solve for the output with each separate input, then add the outputs. This is called superposition.

2. It also tells us that, for networks which are linear, it is not necessary to actually consider the value of the inputs in calculating response. What is important is a system function, or a ratio of output to input.

Superposition is an important principle when dealing with linear networks, and can be used to make analysis easier. If a network has multiple independent sources, it is possible to find the response to each source separately, then add up all of the responses to find total response. Note that this can only be done with independent sources!

Consider, for example, the example circuit shown in Figure 14. If we are only interested in the output voltage \( v_2 \), we may find the response to the voltage source first, then the response to the current source, then the total response is the sum of the two. To find the response to the voltage source, we must “turn off” the current source. This is done by assuming that it is not there. (After all, a current source with zero current is just an open circuit!). The resulting network is as in Figure 15. Note that the resistance \( R_1 \) does not appear here. This is because a resistance in parallel with a voltage source is just a voltage source, unless one is interested in current in the resistance. The output voltage is just:

\[
v_{2v} = V \frac{R_3}{R_2 + R_3}
\]

Next, we “turn off” the voltage source and “turn on” the current source. Note that a voltage source that has been turned off is a short circuit, because that implies zero voltage. The network is as shown in Figure 16 The response to this is:

\[
v_{2i} = IR_2 || R_3
\]

Figure 15: Superposition Fragment: Voltage Source

Figure 16: Superposition Fragment: Current Source
The total response is then just:

\[ v_2 = v_{2v} + v_{2i} = V \frac{R_3}{R_2 + R_3} + I \frac{R_2 R_3}{R_2 + R_3} \]

8 Thevenin and Norton Equivalent Circuits

A particularly important ramification of the property of linearity is expressed in the notion of equivalent circuits. To wit: if we are considering the response of a network at any given terminal pair, that is a pair of nodes that have been brought out to the outside world, it follows from the properties of linearity that, if the network is linear, the output at a single terminal pair (either voltage or current) is the sum of two components:

1. The response that would exist if the excitation at the terminal pair were zero and
2. The response forced at the terminal pair by the exciting voltage or current.

This notion may be expressed with either voltage or current as the response. These yield the Thevenin and Norton equivalent networks, which are exactly equivalent. At any terminal pair, the properties of a linear network may be expressed in terms of either Thevenin or Norton equivalents. The Thevenin equivalent circuit is shown in Figure 17, while the Norton equivalent circuit is shown in Figure 18.

![Thevenin Equivalent Network](image)

Figure 17: Thevenin Equivalent Network

![Norton Equivalent Network](image)

Figure 18: Norton Equivalent Network

A perhaps remarkable fact about the Thevenin and Norton equivalent networks is that they have the same impedance. Further, the equivalent sources are related by the simple relationship:

\[ V_{Th} = R_{eq} I_N \]  

(13)
The *Thevenin Equivalent Voltage*, the source internal to the Thevenin equivalent network, is the same as the *open circuit* voltage, which is the voltage that would appear at the terminals of the equivalent circuit were it to be open circuited. Similarly, the *Norton Equivalent Current* is the same as minus the *short circuit* current.

To consider how we might use these equivalent networks, consider what would happen if the Wheatstone bridge were connected by some resistance across its output, as shown in Figure 19.

![Figure 19: Wheatstone Bridge With Output Resistance](image)

The analysis of this situation is simplified substantially if one recognizes that each side of the bridge can be expressed as either a Thevenin or Norton equivalent network. We may proceed to solve the problem by finding the equivalent networks for each side, then paste them together to form the whole solution. So: consider the equivalent network for the left-hand side of the network, formed by the elements $V$, $R_1$ and $R_2$. This is shown in Figure 20.

![Figure 20: Construction of Equivalent Circuit](image)

Where, here, the components of the equivalent circuit are:

\[
\begin{align*}
V_{Thl} & = V \frac{R_2}{R_1 + R_2} \\
R_{eql} & = R_1 || R_2
\end{align*}
\]

Similarly, the right side of the network is found to have an equivalent source and resistance:

\[
\begin{align*}
V_{Thr} & = V \frac{R_4}{R_3 + R_4} \\
R_{eqr} & = R_3 || R_4
\end{align*}
\]

And the whole thing behaves as the equivalent circuit shown in Figure 21. This is, of course, easily
solved for the current through, and hence the voltage across, the resistance $R_5$, which was desired in the first place:

$$v_5 = \left( v_{Thl} - v_{Thr} \right) \frac{R_5}{R_5 + r_{eql} + r_{eqr}}$$

9 Two Port Networks

So far, we have dealt with a number of networks which may be said to be one port or single-terminal-pair circuits. That is, the important action occurs at a single terminal pair, and is characterized by an impedance and by either a open circuit voltage or a short circuit current, thus forming either a Thevenin or Norton equivalent circuit. A second, and for us very important, class of electrical network has two (or sometimes more) terminal pairs. We will consider formally here the two port network, illustrated schematically in Figure 22.

![Two-Port Network Diagram]

Figure 22: Two-Port Network

There are a number of ways of characterizing this type of network. For the time being, consider that it is passive, so that there is no output without some input. Then we may characterize the network in terms of the currents at its terminals in terms of the voltages, or, conversely, we may describe the voltages in terms of the currents at the terminals. These two ways of describing the network are said to be the admittance or impedance parameters. These may be written in the following way:

The impedance parameter point of view would yield, for a resistive network, the following relationship between voltages and currents:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

(14)
Similarly, the admittance parameter point of view would yield a similar relationship:

\[
\begin{bmatrix}
    i_1 \\
    i_2
\end{bmatrix}
= \begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix}
\begin{bmatrix}
    v_1 \\
    v_2
\end{bmatrix}
\]  

(15)

These two relationships are, of course, the inverses of each other. That is:

\[
\begin{bmatrix}
    G_{11} & G_{12} \\
    G_{21} & G_{22}
\end{bmatrix}
= \begin{bmatrix}
    R_{11} & R_{12} \\
    R_{21} & R_{22}
\end{bmatrix}^{-1}
\]  

(16)

If the networks are linear and passive (i.e. there are no dependent sources inside), they also exhibit the property of reciprocity. In a reciprocal network, the transfer impedance or transfer admittance is the same in both directions. That is:

\[
\begin{align*}
R_{12} &= R_{21} \\
G_{12} &= G_{21}
\end{align*}
\]  

(17)

It is often useful to express two-port networks in terms of T or Π networks, shown in Figures 23 and 24.

![Figure 23: T Equivalent Network](image-url)

![Figure 24: Π Equivalent Network](image-url)

Sometimes it is useful to cascade two-port networks, as is shown in Figure 25. The resulting combination is itself a two-port. Suppose we have a pair of networks characterized by impedance parameters:

\[
\begin{bmatrix}
    v_1 \\
    v_2
\end{bmatrix}
= \begin{bmatrix}
    R_{11} & R_{12} \\
    R_{12} & R_{22}
\end{bmatrix}
\begin{bmatrix}
    i_1 \\
    i_2
\end{bmatrix}
\]
Figure 25: Cascade of Two-Port Networks

\[
\begin{bmatrix}
  v_3 \\
  v_4 
\end{bmatrix} = \begin{bmatrix}
  R_{33} & R_{34} \\
  R_{31} & R_{44}
\end{bmatrix} \begin{bmatrix}
  i_3 \\
  i_4 
\end{bmatrix}
\]

By noting that \( v_2 = v_3 \) and \( i_3 = -i_2 \), it is possible to show, with a little manipulation, that:

\[
\begin{bmatrix}
  v_1 \\
  v_4 
\end{bmatrix} = \begin{bmatrix}
  R'_{11} & R_{14} \\
  R_{14} & R'_{44}
\end{bmatrix} \begin{bmatrix}
  i_3 \\
  i_4 
\end{bmatrix}
\]

where

\[
R'_{11} = R_{11} - \frac{R_{12}^2}{R_{22} + R_{33}} \\
R'_{44} = R_{44} - \frac{R_{34}^2}{R_{22} + R_{33}} \\
R_{14} = \frac{R_{12}R_{34}}{R_{22} + R_{33}}
\]

10 Inductive and Capacitive Circuit Elements

So far, we have dealt with circuit elements which have no memory and which, therefore, are characterized by instantaneous behavior. The expressions which are used to calculate what these elements are doing are algebraic (and for most elements are linear too). As it turns out, much of the circuitry we will be studying can be so characterized, with complex parameters.

However, we take a quick diversion to discuss briefly the transient behavior of circuits containing capacitors and inductors.

![Capacitor and Inductor Symbols](Image)

Figure 26: Capacitance and Inductance

Symbols for capacitive and inductive circuit elements are shown in Figure 26. They are characterized by the relationships between voltage and current:

\[
i_c = C \frac{dv_c}{dt} \quad v_L = L \frac{di_l}{dt}
\]

(18)
Note that, while these elements are linear, since time derivatives are involved in their characterization, expressions describing their behavior in networks will become ordinary differential equations.

10.1 Simple Case: R-C

![Circuit Diagram]

Figure 27: Simple Case: R-C

Figure 27 shows a simple connection of a resistance and a capacitance. This circuit has only two nodes, so there is a single voltage \( v \) across both elements. The two elements produce the constraints:

\[
i_r = \frac{v}{R} \\
i_c = \frac{dv}{dt}
\]

and, since \( i_r = -i_c \),

\[
\frac{dv}{dt} + \frac{1}{RC} v = 0
\]

Now, we know that this sort of first-order, linear equation is solved by:

\[ v \sim e^{-\frac{t}{RC}} \]

(To confirm this, just substitute the exponential into the differential equation.) Then, if we have some initial condition, say \( v(t = 0) = V_0 \), then

\[ v = V_0 e^{-\frac{t}{RC}} \]

This was a trivial case, since we don’t describe how that initial condition might have taken place. But consider a closely related problem, illustrated in Figure 28.

10.2 Simple Case with Drive

The analysis of this circuit is accomplished by noting that it contains a single loop, and adding up the voltages around the loop we find:

\[
RC \frac{dv_c}{dt} + v_c = v_s
\]

Now, assume that the voltage source is a step:

\[ v_s = V_s u_{-1}(t) \]
We should define the step function with some care, since it is of quite a lot of use. The step is one of a hierarchy of singularity functions. It is defined as:

$$ u_{-1}(t) = \begin{cases} 
0 & t < 0 \\
1 & t > 0 
\end{cases} \quad (19) $$

Now, remembering that differential equations have particular and homogeneous solutions, and that for $t > 0$ a particular solution which solves the differential equation is:

$$ v_{cp} = V $$

Of course this does not satisfy the initial condition which is that the capacitance be uncharged: $v_c(t = 0+) = 0$. Again, remember that the whole solution is the sum of the particular and a homogeneous solution, and that the homogeneous solution is the un-driven case. To satisfy the initial condition, the homogeneous solution must be:

$$ c_{ch} = -Ve^{-\frac{t}{RC}} $$

So that the total solution is simply:

$$ v_c = V \left(1 - e^{-\frac{t}{RC}}\right) $$

Next, suppose $v_s = u_{-1}(t)V \cos \omega t$. We know the homogeneous solution must be of the same form, but the particular solution is a bit more complicated. In later chapters we will learn how to make the process of extracting the particular solution easier, but for the time being, let’s assume that with a sinusoidal drive we will get a sinusoidal response of the same frequency. Thus we will guess

$$ v_{cp} = V_{cp} \cos (\omega t - \phi) $$

The time derivative is

$$ \frac{dv_{cp}}{dt} = \omega V_{cp} \sin (\omega t - \phi) $$

so that we can find an algebraic equation for the particular solution:

$$ V \cos \omega t = V_{cp} \left(\cos (\omega t - \phi) + \omega RC \sin (\omega t - \phi)\right) $$

Note the trigonometric identities:

$$ \cos (\omega t - \phi) = \cos \phi \cos \omega t + \sin \phi \sin \omega t $$
$$ \sin (\omega t - \phi) = -\sin \phi \cos \omega t + \cos \phi \sin \omega t $$

16
Since the sine and cosine terms are orthogonal, we can equate coefficients of sine and cosine to get:

\[ V = V_c \left[ \cos \phi + \omega RC \sin \phi \right] \]
\[ 0 = V_c \left[ \sin \phi + \omega RC \cos \phi \right] \]

The second of these can be solved for the phase angle:

\[ \phi = \tan^{-1} \omega RC \]

and squaring both equations and adding:

\[ V^2 = V_c^2 \left( 1 + (\omega RC)^2 \right) \]

so that the particular solution is:

\[ v_{cp} = \frac{V}{\sqrt{1 + (\omega RC)^2}} \cos (\omega t - \phi) \]

Finally, if the capacitor is initially uncharged \((v_c(t = 0+) = 0)\), we can add in the homogeneous solution (we already know the form of this), and find the total solution to be:

\[ v_{cp} = \frac{V}{\sqrt{1 + (\omega RC)^2}} \left[ \cos (\omega t - \phi) - \cos \phi e^{-\frac{t}{\tau}} \right] \]

10.3 Second-Order System Example

Figure 29: Two-Inductor Circuit

Figure 29 shows a network with two inductances and two resistances. Assume that this is driven by a voltage step: \(v_s = V_s u(t)\). Note that, with two inductances, we will require two initial conditions to complete the solution.

The steady state (particular) solution is \(v_o = 0\). There will, of course, be current flowing in each of the inductances, but if excitation is constant there will be no time derivative of current so that voltage across each of the inductances will eventually fall to zero.

The initial conditions may be found by inspection. Right after \(t = 0\) (note we use \(t = 0+\) for this), output voltage must be:

\[ v_o(t = 0+) = V_s \]
This must be so since current cannot be made to flow instantaneously in either inductance, so that there is no current in either resistance.

The second initial condition is the rate of change of voltage right after the instant of the voltage step. To find this, note that output voltage is equal to the source voltage minus the voltage drops across each of the two resistances.

\[ v_o = v_s - R_2 \dot{i}_2 - R_1 (i_1 + i_2) \]

If we differentiate this with respect to time and note that the time derivative of a constant (after the step the input voltage is constant) is zero:

\[ \frac{dv_o}{dt} (t = 0+) = -(R_1 + R_2) \frac{di_2}{dt} - R_1 \frac{di_1}{dt} \]

Now, since right after the instant of the step both inductances have the source voltage \( V_s \) across them:

\[ \frac{di_1}{dt} |_{t=0^+} = \frac{di_2}{dt} |_{t=0^+} = \frac{V_s}{L} \]

the rate of change of voltage at \( t = 0^+ \) is:

\[ \frac{dv_o}{dt} |_{t=0^+} = -\frac{2R_1 + R_2}{L} V_s \]

Now, we can find the homogeneous solution using the loop method. Setting the source to zero, assume a current \( i_a \) in the left-hand loop and \( i_b \) in the right-hand loop. KVL around these two loops yields:

\[ R_1 i_a + L \frac{d}{dt} (i_a - i_b) = 0 \]
\[ R_2 i_b + 2L \frac{di_b}{dt} - L \frac{di_a}{dt} = 0 \]

With a little manipulation, these become:

\[ L \frac{di_a}{dt} + 2R_1 i_a + R_2 i_b = 0 \]
\[ L \frac{di_b}{dt} + R_1 i_a + R_2 i_b = 0 \]

Assume that solutions are of the form \( I e^{st} \), and this set of simultaneous equations becomes:

\[
\begin{bmatrix}
(sL + 2R_1) & R_2 \\
R_1 & (sL + R_2)
\end{bmatrix}
\begin{bmatrix}
I_a \\
I_b
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

We need to solve this for \( s \) (to find values of \( s \) for which this set is true, and that is simply the solution of the “characteristic equation”

\[(sL + 2R_1) (sL + R_2) - R_1 R_2 = 0\]

which is the same as:

\[ s^2 + \frac{2R_1 + R_2}{L} s + \frac{R_1 R_2}{L} = 0 \]
Now, for the sake of “nice numbers”, assume that \( R_1 = 2R \), \( R_2 = 3R \). The characteristic

equation is:

\[
s^2 + 7 \frac{R}{L} s + 6 \left( \frac{R}{L} \right)^2 = 0
\]

which factors nicely into \((s + \frac{R}{L})(s + 6\frac{R}{L}) = 0\), or the two values of \( s \) are \( s = -\frac{R}{L} \) and \( s = -6\frac{R}{L} \).

Since the particular solution to this one is zero, we have a total solution which is:

\[
v_o = Ae^{-\frac{R}{L}t} + Be^{-6\frac{R}{L}t}
\]

The initial conditions are:

\[
v_o|_{t=0^+} = A + B = V_s
\]

\[
\frac{dv_o}{dt}|_{t=0^+} = -\frac{R}{L} (A + 6B) = -7\frac{R}{L}V_s
\]

The solution to that pair of expressions is:

\[
A = -\frac{V_s}{5} \quad B = \frac{6V_s}{5}
\]