Supplemental Notes

Today (Monday, October 16), we didn’t quite get to show that the field lines for two infinite oppositely-charged paraxial cylinders are circles. The way to do this is to write the equation for the equipotentials so that the equipotentials are level sets (surfaces if we consider the z-direction, contours if we restrict ourselves to the plane. In any event, the field lines will be circles.)

So, define a function \( g(x, y) \) by

\[
g(x, y) = k^2 = \frac{r_1^2}{r_2^2} = \frac{(x - a)^2 + y^2}{(x + a)^2 + y^2}.
\]

We have seen that the equipotentials are circles centered on the x-axis at \( x = \pm a \frac{k^2 + 1}{k^2 - 1} \) and radius \( \frac{2a}{k^2 - 1} \).

Anyway, you can show (okay, it took me a while) that

\[
\frac{\partial g}{\partial x} = 4a \frac{(x - a)^2 - y^2}{r_2^4}, \quad \frac{\partial g}{\partial y} = 8a \frac{xy}{r_2^4}.
\]

We now wish to express the curve of the field lines by \( y \) in terms of \( x \). (We could look for some \( y(x) \) explicitly, or \( y \) as an implicit function of \( x \). In this case, it turns out not to matter.) The relation we need is from physics; the field lines will be tangent to the gradient of \( g \) (note that \( g(x, y) \) in not the potential, but the gradient of \( g \) is still tangent to the field lines). Thus,

\[
\frac{dy}{dx} = \frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial x}} = \frac{2xy}{x^2 - a^2 - y^2}.
\]

There are many ways to solve such an item; the way presented below is one of the most direct, but it does use some knowledge of 18.03 and a bit of 18.02 that may not have been presented. That is, we rewrite the above as

\[
2xy \, dx - (x^2 - a^2 - y^2) \, dy = 0,
\]

and see if (remember, that’s an if at this stage) this can be rewritten as

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy = 0.
\]

Well, it can’t. If it could, the term \( 2xy \) would be \( \frac{\partial f}{\partial x} \) so that \( \frac{\partial^2 f}{\partial y \partial x} = 2x \), whereas the term \( -(x^2 - a^2 - y^2) \) would be \( \frac{\partial f}{\partial y} \) so that \( \frac{\partial^2 f}{\partial x \partial y} = -2x \); close but no cigar.
So, what we do is look for an integrating factor; the details of how to do this take a while and don’t add much to the physics (and they’re hard to type), so I’ll just say that we multiply by a factor of $y^{-2}$, and this works, in that we now have

$$\frac{2x}{y} \, dx - \left( \frac{x^2}{y^2} - \frac{a^2}{y^2} - 1 \right) \, dy = 0.$$ 

Fill in the details yourself, but the result is

$$\frac{2x}{y} = \frac{\partial}{\partial x} \left( \frac{x^2}{y} - \frac{a^2}{y} + y \right) - \left( \frac{x^2}{y^2} - \frac{a^2}{y^2} - 1 \right) = \frac{\partial}{\partial y} \left( \frac{x^2}{y} - \frac{a^2}{y} + y \right),$$

and so the field lines follow a curve given by

$$\left( \frac{x^2}{y} - \frac{a^2}{y} + y \right) = c,$$

where $c$ is a constant of integration. Rearranging gives

$$x^2 + (y - c/2)^2 = a^2 + c^2/4.$$

In this form, remember that $c$ could be positive or negative; this might help you decide what the circles look like.