8.02 - ESG Independent Study Fall 2004

Requisite Math for 8.02 Independent Study

By design, the level of mathematics needed for **8.02IS** is higher than that used in 8.02. The presumption is that students in **8.02IS** have successfully completed a course in Newtonian Mechanics (8.01 or the equivalent), Single-variable calculus (18.01 or the equivalent) and have completed or are concurrently taking Multivariable calculus (18.02 or the equivalent).

Some material from 18.02 may be presented before it is encountered in the calculus class. In these cases (partial derivatives, fluxes, work integrals) the material will be presented as part of **8.02IS**, and in a simple manner that will allow application without relying on a deep understanding of the purely mathematical aspects. Indeed, having seen a gradient in a physics class should help understanding of directional derivatives in a math class.

Often in **8.02IS**, solutions of simple differential equations will be needed. The simplest, involving "separable equations," will be considered part of **8.02IS**, in that such equations should have been encountered in 18.01 and problems with similar solutions should have been encountered in 8.01. For more complicated differential equations, it is expected that a given expression can be verified as a solution, but derivation of these solutions is not required. Familiarity with the solutions to the equations governing simple harmonic motion is assumed.

The following list of topics is not exhaustive, but is meant to be indicative. For each, a sample question is given.

- **Differentiation**

Single-variable differentiation of all types of elementary functions will be needed. The chain rule is used extensively. Try this one: If

$$f(x) = \frac{1}{\sqrt{1 + (x - (1/x))^2}},$$

find the extreme values of $f(x)$, $x\, f(x)$ and $f(x)/x$.

- **Limits, Taylor Series, l'Hôpital's Rule**

If you did the above problem, you might have noticed something funny about the ratio $f(x)/x$ near $x = 0$. Or maybe not. In any event, try this one: Let

$$g(x) = \frac{1}{x}\frac{1}{\sqrt{1 + (x - (1/x))^2}}, \qquad x \neq 0.$$

Find $\lim_{x \to 0} g(x)$ and the quadratic (x^2) term of the Taylor Series of $g(x)$ about $x = 0$.

• Integration

Nothing bizarre is needed for **8.02IS**, depending on how one defines "bizarre." The Fundamental Theorem of Calculus is considered, well, fundamental. Integration by parts and use of trigonometric substitutions will occur frequently. Here's a typical integral encountered in **8.02IS**:

$$\int \frac{dx}{(1+x^2)^{3/2}}.$$

• Separable Differential Equations

Suppose an object of mass m is subject to gravity and a frictional force with magnitude proportional to the object's speed. Newton's second law of motion then gives

$$m\,\frac{dv}{dt} = m\,g - b\,v, \qquad \text{or} \qquad \frac{dv}{g - (b/m)\,v} = dt,$$

where the positive direction is down and b is a positive constant. Integrate both sides of the expression on the right and find $v(t)$. Choose constants of integration such that $v(t=0) = 0$. Find $v(t \to \infty)$ and explain why this is what you expect.

• Vector Algebra

This would include vector addition, dot products (scalar products) and cross products (vector products). Here's one that's useful, but sort of tricky because it's purely symbolic.

Given a nonzero vector $\mathbf{A}$ and a unit vector $\hat{\mathbf{n}}$, find vectors $\mathbf{A}_\perp$ and $\mathbf{A}_\parallel$ such that

$$\mathbf{A} = \mathbf{A}_\perp + \mathbf{A}_\parallel, \qquad \mathbf{A}_\perp \cdot \hat{\mathbf{n}} = 0, \qquad \mathbf{A}_\parallel \times \hat{\mathbf{n}} = \mathbf{0}.$$

• Differentiation of Vectors

Really, this is not the same as vector calculus. "Differentiation of Vectors" is what we do to find vector velocities and accelerations. Here's a neat example: If

$$x(t) = e^t\,\cos t, \qquad y(t) = e^t\,\sin t,$$

find the vector velocity $\mathbf{v}(t)$, the vector acceleration $\mathbf{a}(t)$ and the angle between $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

Notation, Notation, Notation

A vector equation which does not use proper vector notation is just plain wrong. To oversimplify matters somewhat, the most common mistakes are:

(1) Neglecting to distinguish between vectors and scalars, and

(2) Neglecting to distinguish between the magnitude of a vector and the respective components of the same vector.

8.02IS, and virtually every physics application currently in use, will use the convention that the magnitude of a vector $\mathbf{A}$ is represented by

$$|\mathbf{A}| = A;$$

the lightface type means that A is a scalar. For non-zero vectors, we will define a unit vector in the direction of $\mathbf{A}$ by

$$\hat{\mathbf{A}} \equiv \frac{\mathbf{A}}{A} \quad \text{so that} \quad \mathbf{A} = A\,\hat{\mathbf{A}}.$$

Those interested in the mathematical details will notice that we have tacitly assumed a crucial axiom of metric spaces, in that $A = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

Consider the familiar form of Newton's second law,

$$\mathbf{F} = m\,\mathbf{a}.$$

The above form tells us that force and acceleration are both vectors and are parallel. By taking magnitudes of both sides, we obtain the even more familiar

$$F = m\,a.$$

The two expressions are both valid, but they are *not* the same. One of the reasons this matters is that if $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, then

$$\mathbf{F}_1 + \mathbf{F}_2 = m\,\mathbf{a}, \qquad \text{but}$$

$$F_1 + F_2 \neq m\,a$$

except in the special case that $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel.

For E&M, the distinction is even more crucial. For instance, the electrostatic force on a particle of charge q is given by

$$\mathbf{F}_{\text{elec}} = q\,\mathbf{E},$$

but since charge, unlike mass, can be either positive or negative (or zero), the correct relation between the magnitudes is

$$F_{\text{elec}} = |q|\ E.$$

For Coulomb's Law, the field at the origin due to two point charges of charges q_1 and q_2 located at postions $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively is

$$\mathrm{E}_{\text{origin}} = -\left(\frac{1}{4\,\pi\epsilon_0}\right)\left[q_1\frac{\mathbf{r}_1}{|r_1|^3} + q_2\frac{\mathbf{r}_2}{|r_2|^3}\right]$$

where $\mathbf{r}_1$ is the vector *from* the origin to the position of the charge q_1 and similarly for $\mathbf{r}_2$. Unless these vectors are colinear, the magnitude of the sum will not be either the scalar sum or scalar difference of the two contributions to the field.

You *must* use some notation to distinguish between vectors and scalars. Handwritten bold (such as $\mathbb{E}$) is hard, but some try (Feynman liked to do this). A common usage in handwritten material is to use overarrows, such as $\vec{\mathbf{E}}$, $\vec{E}$ or $\overrightarrow{\mathrm{E}}$, the advantage to the last being that the big overarrow clearly indicates a vector, regardless which style is used for the character.

For all typeset notes and tests in this subject, bold characters will be used for vectors, and unit vectors will have the caret, as in $\hat{\mathbf{x}}$. Subscripts will indicate a component. For instance, the vector $\mathbf{E}$ in component form would be

$$\mathbf{E} = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}} + E_z\,\hat{\mathbf{z}}.$$

That is, $\hat{\mathbf{x}}$ will be used instead of $\hat{\mathbf{i}}$, $\hat{\boldsymbol{\imath}}$, $\hat{i}$ or $\hat{\imath}$, and similarly for other unit coordinate vectors.