8.02 - ESG Independent Study Spring 2005
Unit 5: Mathematical Appendix

Those with experience in 18.03 techniques or theory may recognize that since the separated equation for $f_n(\theta)$, given in Unit 5 as

$$0 = n(n+1)f_n + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta}f_n\right),$$

is a second order linear ordinary differential equation, and should have two linearly independent solutions in any interval not including $\sin\theta = 0$, in our case $0 < \theta < \pi$. The form of the solution presented, that is, a polynomial of degree n for integer $n \geq 0$, seems to disdain the other solution. To cut to the chase, the other solution can be shown to have an infinite power series solution which diverges as $\sin\theta \to 0$, and so is unacceptable.

For more detailed sources on the following, consult any Differential Equations text; check the index under "Legendre's Equation" or "Legendre Polynomials." Another good source is Jackson's "Classical Electrodynamics"; the page numbers change with edition, but either check the index or the back endpapers.

Before getting fancy, let's "do the drill" in the simple case $n = 0$, in which case we find, more or less by inspection, a solution $f_{0,1} \equiv 0$ (the extra subscript will be 1 or 2, just to distinguish two indpendent solutions). Let's do better, and integrate the above equation (with $n = 0$) to find

$$\begin{aligned} \sin\theta\,\frac{d}{d\theta}f_n &= -c_2 \\ \frac{d}{d\theta}f_n &= -c_2\csc\theta \\ f_n &= c_2\ln(\csc\theta + \cot\theta) + c_1. \end{aligned}$$

Note that the names given the constants of integration were seriously rigged so that $c_1 = 1$ corresponds to $f_{0,1}$, and we can take $c_2 = 1$ to correspond to $f_{0,2}$. Although it doesn't matter right now, it will help later to note that

$$f_{0,2} = \ln(\csc\theta + \cot\theta) = \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right).$$

This shows us two things: First, the argument of the square root, and hence the logarithm, is never negative, so we don't have to worry about absolute value signs. Next, the solution clearly diverges at $\cos\theta = \pm 1$, and so is not part of our physical solution.

The above procedure gets nasty for integer $n > 0$, though. For instance, for $n = 1$, $f_{1,1} = \cos\theta$, use the standard "Reduction of Order" technique; that is, let $f_{1,2} = v(\theta)\, f_{1,1} = v(\theta)\,\cos\theta$, insert into equation, turn crank, cross fingers and arrive at

$$\frac{v''(\theta)}{v'(\theta)} = \frac{d}{d\theta}\ln(v'(\theta) = 2\,\tan\theta - \cot\theta \qquad \longrightarrow \qquad v'(\theta) = \sec^2\theta\,\csc\theta$$

if done correctly (hey, it took me a few tries). The primes denote differentiation with respect to θ. Play with this last expression at your leisure; I know of no way to do the needed integral that does not involve integration by parts. The result is $f_{1,2} = 1 - \frac{\cos\theta}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$, which has similar difficulties at $\cos\theta = \pm 1$.

Before going on, note that $f_{0,1}$ is a (trivially) even function of $\cos\theta$, $f_{0,2}$ is an odd function of $\cos\theta$, $f_{1,2}$ is an odd function of $\cos\theta$ and $f_{1,2}$ is an even function of $\cos\theta$. Indeed, this pattern can be shown to be valid for all integer $n \geq 0$.

To look at these functions further and a bit more rigorously, it helps to make the change of dependent variable $x = \cos\theta$; indeed, this substitution is almost the only way to do the above integral to find v from v'. The differential equation for f_n then becomes

$$n(n+1)\,f_n + \frac{d}{dx}\left(\left(1 - x^2\right)\frac{d}{dx}f_n\right) = 0.$$

In this form, the usual methods of obtaining series solutions ("Frobenius Solutions") may be employed. The technique won't be reproduced in these notes, but it can be seen that near $x = 0$ ($\cos\theta = 0$, $\theta = \pi/2$), two independent Frobenius Series solutions exist, one with lowest power x^0 and one with lowest power x^1, *regardless of the value of* n. It is further seen, however, that if n is an integer, one series will truncate after the x^n term, and the other will have an infinite series expression which diverges logarithmically at $x \to \pm 1$.

The reason for presenting this in this appendix is to emphasize that the two solutions are *not* found by considering the problem as an "exceptional case;" the roots of the indicial equation, 0 and 1, while differing by an integer, both lead to solutions valid for $-1 < x < 1$, but we discard the divergent solutions if we want results physically valid for the whole sphere.

Warning: Quantum Mechanics is, of course, different. Sometimes we want and even seek divergent solutions.