8.02 - ESG Independent Study Spring 2005
Unit 5: Conductors

This unit is a short one, intended to provide a link between the so far theoretical discussion of electric fields and capacitors, which are actual everyday electrical components. Also, we introduce a part of potential theory, which has applications to many topics (besides electromagnetism), including heat conduction and fluid flow.

The material for this unit is not covered rigorously in UP11 (the applicable section is Section 21.2). For this unit, you should read in Purcell (second edition; ask if you want to borrow a copy of the first edition) Chapter 3, sections 1–4 & 8. The discussion is not numerical, and should be understood by those unfamiliar with Gaussian units. If you're used to MKSA, you might even want to rederive Purcell's equations in these sections using MKSA units (once you do two or three at the most, you'll appreciate how simple conversion is.). Do problems 2, 3, 5, 7 and 8 from Chapter 3. Problem 3.3 has a very elegant solution and is immensely instructive.

The unit "test" for this unit consists of doing the following problem and *explaining* your result to a tutor. That is, besides writing a clear and concise solution, you should be prepared to defend your work to a not-so-skeptical audience. You might be asked to find and demonstrate another aspect of your result. Note that in the problem, electrostatic potential is represented as ψ (psi) to distinguish it from the angular variable ϕ.

The needed vector derivatives for this unit are in the last page of these guidelines. Or, you may download a more extensive table from the **8.02IS-ESG** web page.

In addition, those familiar with differential equations at the 18.03 level might spot a place where we've tried to sweep some upleasantness under a rug, but found no rug. The details are not important to this unit, and so are given in a **Mathematical Appendix to Unit 5**, linked from the web page but not attached to this unit description.

Test Problem

A uniform perfectly conducting sphere of radius R is placed in a uniform electric field $\mathbf{E}_\mathrm{a}$, of magnitude E_a (the subscript "a" is for "ambient"). What is the induced dipole moment of the sphere?

Suggested procedure:

(1) Set up a spherical polar coordinate system with origin at (where else?) the center of the sphere. Since the sphere is a conductor, its surface is an equipotential, so let the potential (ψ) at the surface be zero. Let the $\theta = 0$ direction be the direction of the uniform field, and express ψ_0, the potential corresponding to the uniform field, in terms of E_a, r, and θ.

The table at the end of these guidelines is taken from Jackson, *Classical Electrodynamics*, and it's a good thing to have. Please be careful; here,

$$x = r\,\sin\theta\,\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\,\sin\theta\,\sin\phi \qquad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = r\,\cos\theta \qquad \tan\phi = \frac{y}{x}.$$

Your math instructor may have told you something specious.

(2) Find the new potential outside the sphere. To do this, assume that the potential is a sum of terms of the form $r^n f_n(\theta)$ (no ϕ dependence). Specifically, assume

$$\psi = \sum_n A_n\, r^n\, f_n(\theta),$$

where the A_n are constants. Using the attached table, derive a simple differential equation for f_n in terms of n and θ that must be satisfied so that each term in the above proposed sum satisfies the needed condition for the potential outside the sphere. That is, show that

$$0 = n(n+1) f_n + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta} f_n\right).$$

NOTE: This (item 3) is where those with 18.03 knowledge may spot something fishy. See the **Mathematical Appendix to Unit 5** for a non-apology.

(3) Show that (within possible multiplicative constants), $f_0 = f_{-1}$, $f_1 = f_{-2}$, $f_2 = f_{-3}$, etc. Find f_n for $n = -3$ to 2. Use the fact (for now, take it as given) that if $n \geq 0$ is an integer, f_n is an n^{th} order polynomial in $\cos\theta$.

(4) Now, for the case of the conducting sphere we need

(a) as $r \gg R$, $\psi = \psi_0$

(b) at $r = R$, $\psi = 0$.

Using these, show that the *only* ψ satisfying these conditions consists of *two terms* and is of the form

$$\psi = A\,r^n f_n(\theta) + B\,r^m f_m(\theta).$$

Find A, B, n and m.

(5) In Part 4, you should have found that the presence of the sphere adds a term to the potential corresponding to a dipole field. Since the potential due to a dipole is $(p\cos\theta)/\left(4\pi\epsilon_0 r^2\right)$ in MKSA, or $(p\cos\theta)/r^2$ in CGS, where p is the dipole moment, find p in terms of E_{a} and R.

(6) Find $\mathbf{E}$ and σ (the surface charge density) on the surface of the sphere. (Use the table to find $\mathbf{E}$ from ψ if you need to.) Show that $\int \sigma\,dA = 0$, and find by direct integration

$$\mathbf{p} = \int \mathbf{r}\,\sigma\,dA.$$

The simplest way to do this is to denote

$$\mathbf{r} = \hat{\mathbf{x}}\,r\sin\theta\cos\phi + \hat{\mathbf{y}}\,r\sin\theta\sin\phi + \hat{\mathbf{z}}\,r\cos\theta.$$

On the sphere, $r = R$. To do the integral in spherical coordinates, use $dA = R^2\,\sin\theta\,d\theta\,d\phi$.

Check that $|\mathbf{p}|$ is the same as found in Part 5 above, and so express

$$\mathbf{p} = \alpha\,\mathbf{E}_{\text{a}}.$$

(7) If there are n such spheres per unit volume, the polarizability is then $\chi = n\alpha$. Thus we have a simple model of a dielectric, which we will use in the next unit.

A Partial Table of Vector Operators

in

Spherical Polar Coordinates

Using

PHYSICS Notation

$$x = r\,\sin\theta\,\cos\phi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\,\sin\theta\,\sin\phi \qquad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$z = r\,\cos\theta \qquad \tan\phi = \frac{y}{x}$$

$$0 \le \theta \le \pi, \qquad 0 \le \phi \le 2\,\pi$$

$$\boldsymbol{\nabla}\,\psi = \hat{\mathbf{r}}\,\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\,\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{r\,\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\theta}\,\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\,\psi\right).\right]$$