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IV. Gauss's Law - Worked Examples

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Example 1: Electric flux due to a positive point charge

Consider a positive point charge $Q > 0$ located at a point P. The electric field of this charge is given by

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} \tag{1.1}$$

where $\hat{\mathbf{r}}$ is a unit vector located at the point P, that points from Q to the point P. What is the flux of the electric field on a sphere of radius r centered on Q?

Solution:

There are two important things to notice about this electric field. The first point is that the electric field is constant in magnitude on a sphere of radius r centered on the charge Q. The second point is that the electric field points radially away from Q.

Let's calculate the flux of the electric field on a sphere of radius r centered on Q. First we choose a small patch of that sphere of radius ΔA_i.

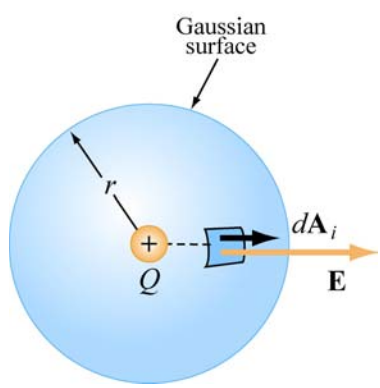

Figure 1.1 Flux of a point charge on a sphere

Since the electric field at each point on the sphere points outward from the center of the sphere, it is perpendicular to the plane of the patch. So the electric flux through this patch is

$$(\Delta\Phi_E)_i = E\Delta A_i = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\Delta A_i \tag{1.2}$$

The total flux through all the patches is just the sum,

$$\Phi_E = \lim_{N\to\infty} \sum_{i=1}^{i=N} E\Delta A_i \equiv \oint\!\!\oint_S E\,dA \tag{1.3}$$

The crucial point to notice is that the electric field is constant on the sphere. This means that the electric field can be pulled in front of the summation sign, (or equivalently outside the integration)

$$\Phi_E = E\oint\!\!\oint_S dA \tag{1.4}$$

The summation of the small area elements over all the patches is just the total area of the sphere

$$\oint\!\!\oint_S dA = A = 4\pi r^2 \tag{1.5}$$

So the flux is just the product of the magnitude of the electric field with the area of the sphere, (similar to the calculation for a constant field on a square area),

$$\Phi_E = EA = \left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\right)4\pi r^2 = \frac{Q}{\varepsilon_0} \tag{1.6}$$

where

$$\varepsilon_0 = \frac{1}{4\pi(9.0\times10^9\ \mathrm{N\cdot m^2/C^2})} = 8.84\times10^{-12}\ \mathrm{C^2/N\cdot m^2} \tag{1.7}$$

is the permitivitty of free space.

This calculation for a point charge will be the basis for Gauss's Law. Notice that the crucial property of the electric field is it's inverse square dependence on distance r. If the field were any other power of r, the flux would no longer equal Q/ε_0 for a point charge and Gauss's Law would not be true!

Example 2: Electric flux through a square surface

Compute the electric flux through a square surface of edges $2l$ due to a charge $+Q$ located at a perpendicular distance l from the center of the square, as shown in Figure 2.1.

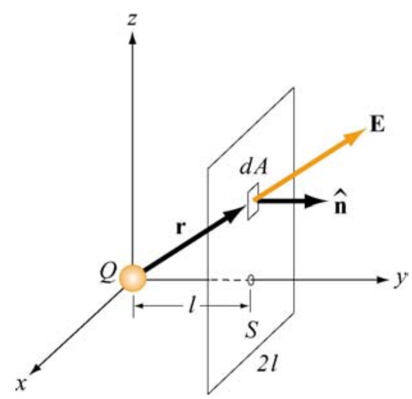

Figure 2.1 Electric flux through a square surface

Solution:

The electric field due to the charge $+Q$ is

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{r}\right) \tag{2.1}$$

where $r=(x^2+y^2+z^2)^{1/2}$ in Cartesian coordinates. On the surface S, $y=l$ and the area element is $d\vec{\mathbf{A}} = dA\hat{\mathbf{j}} = (dxdz)\hat{\mathbf{j}}$. Since $\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0$ and $\hat{\mathbf{j}}\cdot\hat{\mathbf{j}} = 1$, we then have

$$\vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{r}\right)\cdot(dxdz)\hat{\mathbf{j}} = \frac{Ql}{4\pi\varepsilon_0 r^3}dxdz \tag{2.2}$$

Thus, the electric flux through S is

$$\Phi_E = \oiint_S \vec{\mathbf{E}}\cdot d\vec{\mathbf{A}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\int_{-l}^{l}\frac{dz}{(x^2+l^2+z^2)^{3/2}} = \frac{Ql}{4\pi\varepsilon_0}\int_{-l}^{l}dx\left.\frac{z}{(x^2+l^2)(x^2+l^2+z^2)^{1/2}}\right|_{-l}^{l}$$

$$= \frac{Ql}{2\pi\varepsilon_0}\int_{-l}^{l}\frac{l\,dx}{(x^2+l^2)(x^2+2l^2)^{1/2}} = \frac{Q}{2\pi\varepsilon_0}\tan^{-1}\left.\left(\frac{x}{(x^2+2l^2)^{1/2}}\right)\right|_{-l}^{l}$$

$$= \frac{Q}{2\pi\varepsilon_0}\left[\tan^{-1}(1/\sqrt{3}) - \tan^{-1}(-1/\sqrt{3})\right] = \frac{Q}{6\varepsilon_0}$$

$$\boxed{\Phi_E = \frac{Q}{6\varepsilon_0}} \tag{2.3}$$

where the following integrals have been used:

$$\begin{aligned} &\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}} \\ &\int \frac{dx}{(x^2+a^2)(x^2+b^2)^{1/2}} = \frac{1}{a(b^2-a^2)^{1/2}} \tan^{-1} \sqrt{\frac{b^2-a^2}{a^2(x^2+b^2)}}, \quad b^2 > a^2 \end{aligned} \tag{2.4}$$

Example 3: Electric flux through a cube

Place a charge $+Q$ at the center of a cube of side $2l$ (Figure 3.1), what is the total flux emerging from all the six faces of the closed surface?

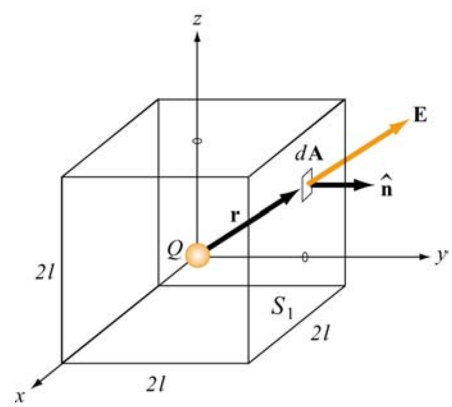

Figure 3.1 Electric flux through the surface of a cube

Solution: From symmetry argument, the flux through each face must be the same. Thus, the total flux through the cube is just six times that through one face:

$$\Phi_E = 6\left(\frac{Q}{6\varepsilon_0}\right) = \frac{Q}{\varepsilon_0} \tag{3.1}$$

The result shows that the electric flux Φ_E passing through a closed surface is proportional to the charge enclosed. In addition, Φ_E is independent of the shape of the closed surface.

Example 4: Non-conducting solid sphere

An electric charge $+Q$ is uniformly distributed throughout a non-conducting solid sphere of radius a. Determine the electric field everywhere inside and outside the sphere.

Solution:

Step 1: The charge distribution is spherically symmetric.

Step 2: Since $+Q$ is uniformly distributed throughout the volume, the electric field $\vec{\mathbf{E}}$ must be radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius r.

Step 3: The charge density of the sphere is uniform and given by

$$\rho = \frac{Q}{V} = \frac{Q}{(4/3)\pi a^3} \tag{4.1}$$

where V is the volume of the sphere. The charge distribution divides space into two regions,

1. $r \le a$
2. $r \ge a$.

Region 1: Consider the first case where $r \le a$.

Step 4a: We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 4.1 below.

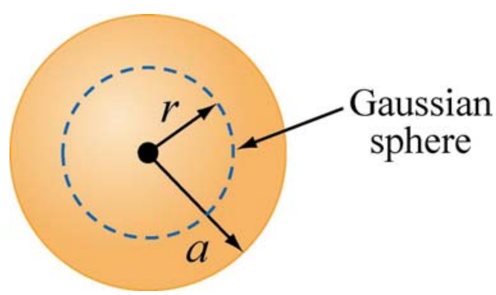

Figure 4.1 Gaussian surfaces for uniformly charged solid sphere with $r \le a$

Step 5a: The flux through the Gaussian surface is

$$\Phi_E = \oint\!\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2) \tag{4.2}$$

Step 6a: The charge distribution is uniform. Therefore, the charge enclosed in the Gaussian sphere of radius r is

$$q_{in} \equiv \int_V \rho dV = \rho V = \rho\left(\frac{4}{3}\pi r^3\right) \tag{4.3}$$

Step 7a: We can now apply Gauss's Law and equate

$$\Phi_E = \frac{q_{in}}{\varepsilon_0} \tag{4.4}$$

This becomes

$$E\left(4\pi r^2\right) = \frac{\rho}{\varepsilon_0}\left(\frac{4}{3}\pi r^3\right) \tag{4.5}$$

We can now solve for the magnitude of the electric field in the region $r \le a$,

$$E = \frac{\rho r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 a^3}, \quad r \le a \tag{4.6}$$

Region 2: We now repeat steps 4 through 7 for the second region, $r \ge a$.

Step 4b: We choose our Gaussian surface to be a sphere of radius $r \ge a$, as shown in Figure 4.2 below.

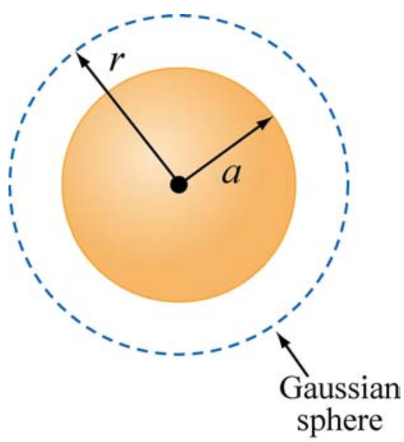

Figure 4.2 Gaussian surfaces for uniformly charged solid sphere with $r \ge a$

Step 5b: The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E(4\pi r^2) \tag{4.7}$$

Notice that the form of our answer for the flux did not change for the case $r \geq a$.

Step 6b: The charge distribution is uniform. Since the radius of the Gaussian surface is greater than the radius of the sphere all the charge is enclosed in our Gaussian surface,

$$q_{in} = \int_V \rho dV = \rho\left(\frac{4}{3}\pi a^3\right) = Q \tag{4.8}$$

This time our answer does change!

Step 7b: Upon applying Gauss's Law, we obtain.

$$E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \tag{4.9}$$

We can now solve for the magnitude of the electric field in the region $r \geq a$,

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > a \tag{4.10}$$

The field outside the sphere is the same as if all the charges were concentrated at the center of the sphere.

Step 8: The qualitative behavior of E as a function of r is plotted in Figure 4.3.

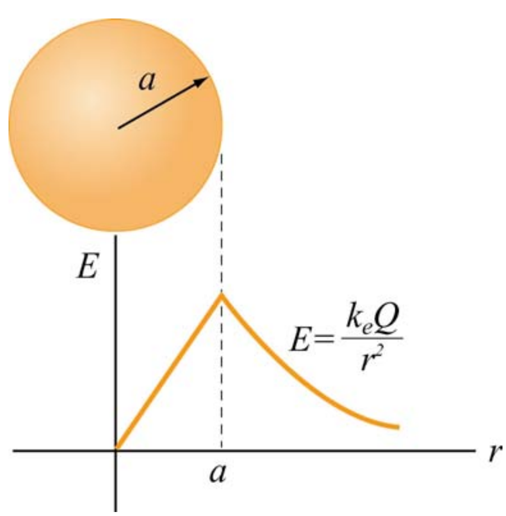

Figure 4.3 Electric field due to a uniformly charged sphere as a function of r

Example 5: Spherical shell

A thin spherical shell of radius a has a charge $+Q$ evenly distributed over its surface. Find the electric field both inside and outside the shell.

Solution:

Step 1: The charge distribution is spherically symmetric.

Step 2: Since $+Q$ is uniformly distributed on the shell, the electric field $\vec{\mathbf{E}}$ must be radially symmetric and directed outward. The magnitude of the electric field is constant on spherical surfaces of radius r.

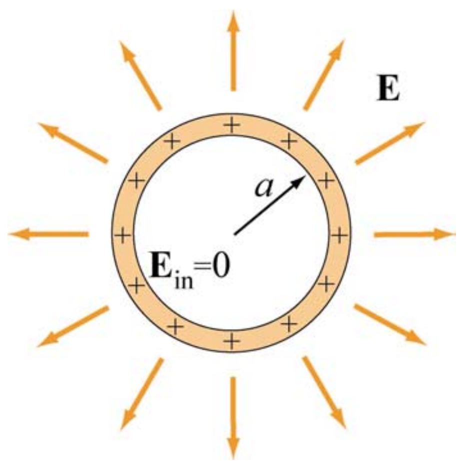

Figure 5.1 Electric field for uniform spherical shell of charge

Step 3: The surface charge density of the sphere is uniform and given by

$$\sigma = \frac{Q}{A} = \frac{Q}{4\pi a^2} \tag{5.1}$$

where A is the surface area of the sphere. The charge distribution divides space into two regions,

3. $r \le a$
4. $r \ge a$.

Region 1: Consider the first case where $r \le a$.

Step 4a: We choose our Gaussian surface to be a sphere of radius $r \le a$, as shown in Figure 5.2 below.

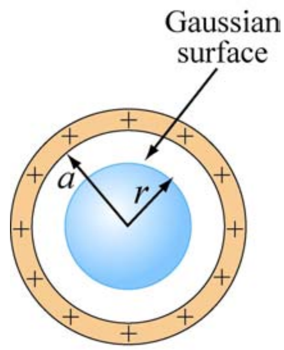

Figure 5.2 Gaussian surfaces for uniformly charged spherical shell with $r \le a$

Step 5a: The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E\left(4\pi r^2\right) \tag{5.2}$$

Step 6a: Since all the charge is located on the shell, the charge enclosed in the Gaussian sphere of radius r is zero,

$$q_{in} = \int_V \rho dV = 0 \tag{5.3}$$

Step 7a: From Gauss's Law, $\Phi_E = q_{in} / \varepsilon_0$, we have $E\left(4\pi r^2\right) = 0$ which implies

$$E = 0, \ \ r < a \tag{5.4}$$

Region 2: We now repeat steps 4 through 7 for the second region, $r \ge a$.

Step 4a: We choose our Gaussian surface to be a sphere of radius $r \ge a$, as shown in Figure 5.3 below.

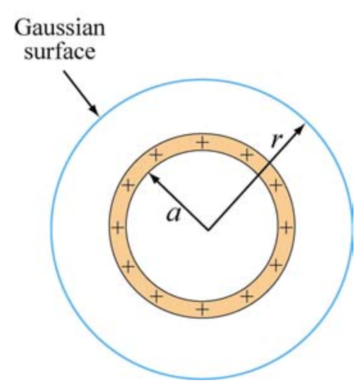

Figure 5.3 Gaussian surfaces for uniformly charged spherical shell with $r \ge a$

Step 5b: The flux through the Gaussian surface again is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E\left(4\pi r^2\right) \tag{5.5}$$

Step 6b: Since the radius of the Gaussian surface is greater than the radius of the sphere all the charge is enclosed in our Gaussian surface,

$$q_{in} = \int_V \rho dV = \rho V = \rho\left(\frac{4}{3}\pi a^3\right) = Q \tag{5.6}$$

Step 7b: We can now apply Gauss's Law $\Phi_E = q_{in} / \varepsilon_0$, which yields

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r \geq a \tag{5.7}$$

The field outside the sphere is the same as if all the charges were concentrated at the center of the sphere just as in the case of the solid sphere with uniform charge density.

Step 8: The qualitative behavior of E as a function of r is plotted in Figure 5.4.

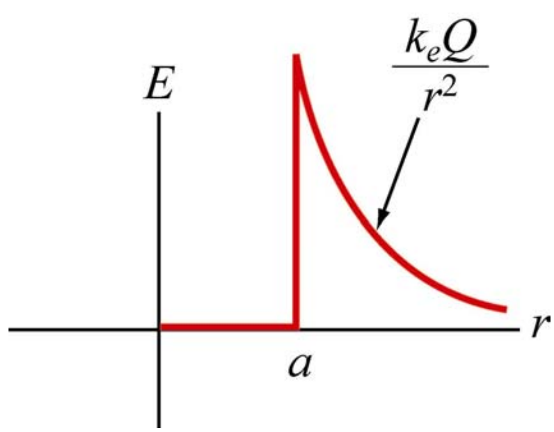

Figure 5.4 Electric field due to a uniformly charged spherical shell as a function of r

Example 6: Gauss's Law for gravity

What is the gravitational field inside a spherical shell of radius a and mass m?

Solution:

Since the gravitational force is also an inverse square law, there is an equivalent Gauss's Law for gravitation,

$$\Phi_g = -4\pi G m_{in} \tag{6.1}$$

The only changes are that we calculate gravitational flux, the constant $1/\varepsilon_0 \rightarrow -4\pi G$, and $q_{in} \rightarrow m_{in}$. For $r \leq a$, the mass enclosed in a Gaussian surface is zero because the mass is all on the shell. Therefore the gravitational flux on the Gaussian surface is zero. This means that the gravitational field inside the shell is zero!

Example 7: Infinitely long rod of uniform charge density

An infinitely long rod of negligible radius has a uniform linear charge density λ. Calculate the electric field at a distance r from the wire.

Solution:

Step 1: The charge distribution is cylindrically symmetric.

Step 2: Since λ is uniformly distributed throughout the volume, the electric field $\vec{\mathbf{E}}$ must be point radially away from the symmetry axis of the rod (Figure 7.1). The magnitude of the electric field is constant on cylindrical surfaces of radius r.

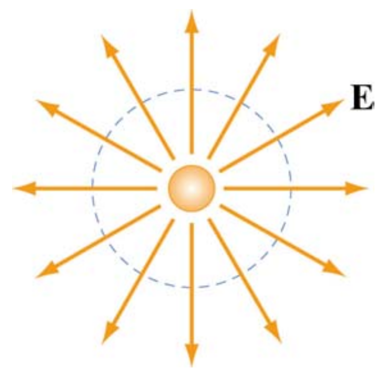

Figure 7.1 Electric field for an infinite uniformly charged rod (symmetry axis of rod is perpendicular to plane of the page)

Step 3:

There is only one region of space since the radius of the rod is negligible.

Step 4: We choose our Gaussian surface to be a cylinder of radius r and length l as shown in Figure 7.22.

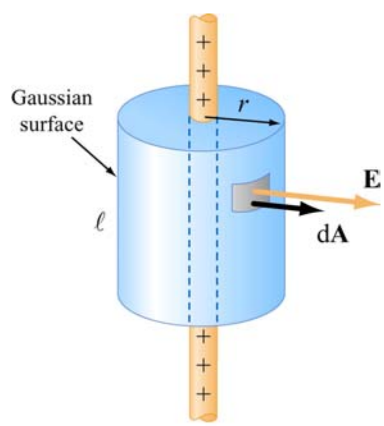

Figure 7.2 Gaussian surfaces for uniformly charged rod

Step 5: The flux through the Gaussian surface is

$$\Phi_E = \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E\left(2\pi rl\right) \tag{7.1}$$

where A is the surface area of the cylinder. Notice that the Gaussian surface is not infinite like the rod but finite and of length l. If we choose our Gaussian surface to be infinite in length, the area of the Gaussian surface would be infinite and hence the flux would also be infinite.

Step 6: The linear charge distribution is uniform. Therefore the charge enclosed in the Gaussian cylinder of radius r and length l is

$$q_{in} = \lambda l \tag{7.2}$$

Step 7: Applying Gauss's Law gives

$$E\left(2\pi rl\right) = \frac{\lambda l}{\varepsilon_0} \tag{7.3}$$

We can now solve for the magnitude of the electric field in the region $r \le a$,

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}} \tag{7.4}$$

Notice that the result for the magnitude of the electric field is independent of the length of the Gaussian surface and only depends on the inverse of the distance r from the symmetry axis.

Step 8: The qualitative behavior of E as a function of r is plotted in Figure 7.3.

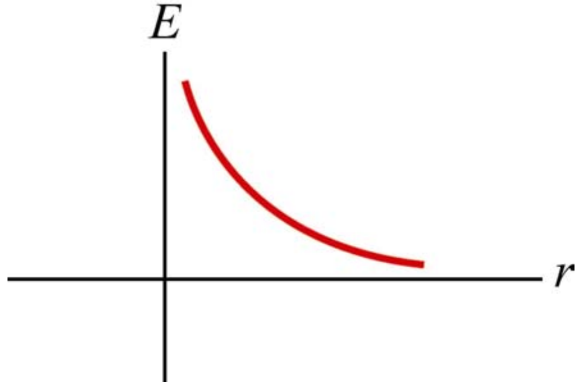

Figure 7.3 Electric field due to a uniformly charged rod as a function of r

Example 8: Infinite plane of charge

Consider an infinitely large non-conducting plane with uniformly positive surface charge density σ. Determine the electric field everywhere in space.

Solution:

Step 1: The charge distribution has a planar symmetry.

Step 2: Since σ is uniformly distributed on the surface, the electric field $\vec{\mathbf{E}}$ must point perpendicularly away from the plane. The magnitude of the electric field is constant on planes parallel to the non-conducting plane.

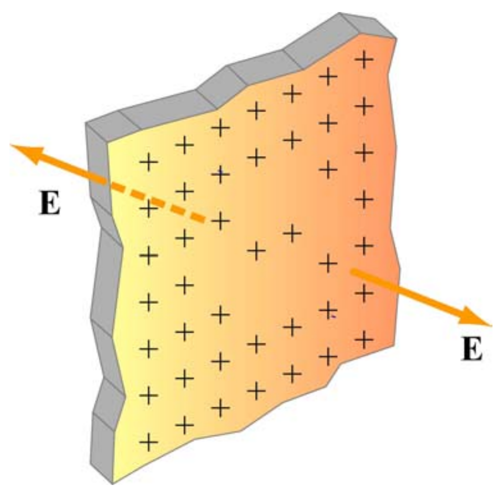

Figure 8.1 Electric field for uniform plane of charge

Step 3: The surface charge density of the plane is uniform and equal to σ.

Let's define an x-coordinate axis with the non-conducting plane located at $x = 0$. The surface charge distribution divides space into two regions,

5. $x > 0$
6. $x < 0$

For the moment, we do not include the non-conducting plane at $x = 0$ in our two regions. This is due to the fact that we assume the plane is of negligible thickness.

Step 4: Although we have two regions of space, we need to choose our Gaussian surface so that some of the surface charge is enclosed. Therefore the Gaussian surface must extend on both sides of the plane. We choose it to be a cylindrical tube as shown in

Figure 8.2 below. The tube has total length l and is evenly divided by the non-conducting plane. The tube has two end-caps, each of area A.

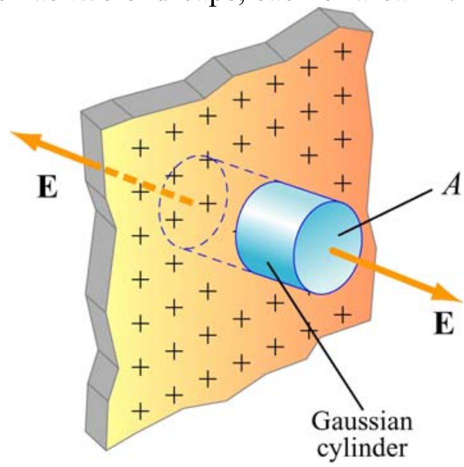

Figure 8.2 Gaussian surface for uniformly charged non-conducting plane

Step 5: The Gaussian surface consists of three faces, S_1 (for end-cap 1), S_2 (for end-cap 2), and the open cylinder S_3. The total flux on S is the sum of the flux on the individual surfaces,

$$\Phi_E \equiv \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \quad (8.1)$$

On the open cylinder the electric field is tangent to the surface, so the flux is zero,

$$\iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 = 0 \quad (8.2)$$

On the end-caps the electric field points outward and since the end-caps are the same distance away from the plane, the magnitude of the electric field on the end-caps are equal,

$$E_1\big|_{x=-l/2} = E_2\big|_{x=l/2} \quad (8.3)$$

Therefore the fluxes on S_1 and S_2 are equal,

$$\iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 = \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 = EA \quad (8.4)$$

So the total flux on the closed Gaussian cylinder is

$$\Phi_E = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 = 2EA \quad (8.5)$$

Step 6: The surface charge distribution on is uniform. The area of the intersection of he non-conducting plane with the Gaussian cylinder is equal to the area of the end-caps, A. Therefore the charge enclosed in the cylinder is $q_{in} = \sigma A$.

Step 7: By applying Gauss's Law, we have

$$2EA = \frac{q_{in}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \tag{8.6}$$

which gives

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}} \tag{8.7}$$

The magnitude of the electric field is independent of the distance of the end-cap from the non-conducting plane. This means that the electric field is uniform in space. However, the electric field does change direction as we move from $x < 0$ to $x > 0$. So our electric field can be expressed in vector form as

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x < 0 \end{cases} \tag{8.8}$$

Step 8: We plot the x-component of the electric field as a function of the variable x. Although the magnitude of the electric field, E, is the same everywhere, we note that there is a discontinuity in the x-component of the electric field at $x = 0$.

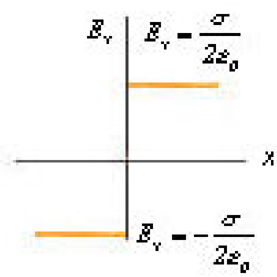

Figure 8.3 Electric field as a function x for infinite plane sheet of charge

Example 9: Electric field of two parallel planes

Consider two parallel infinite non-conducting planes that are separated by a distance d. Each disc is uniformly charged with equal but opposite surface charge densities, $\sigma_+ = \sigma > 0$ and $\sigma_- = -\sigma < 0$.

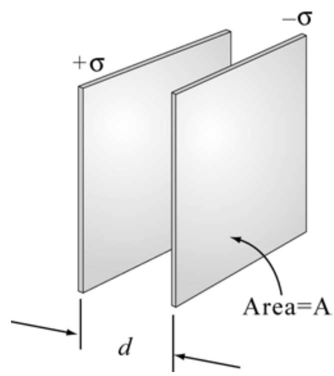

Figure 9.1 Positive and negative uniformly charged infinite planes

In order to find the total electric field everywhere in space we must add together the electric fields of the two oppositely charged planes (the superposition principle).

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_+ + \vec{\mathbf{E}}_- \tag{9.1}$$

The magnitudes of both fields are equal, because the planes carry equal but opposite surface charge densities,

$$E_+ = E_- = \frac{\sigma}{2\varepsilon_0} \tag{9.2}$$

The field of the positive plane points away from the positive plane and the field of the negative plane points towards the negative plane (Figure 9.2).

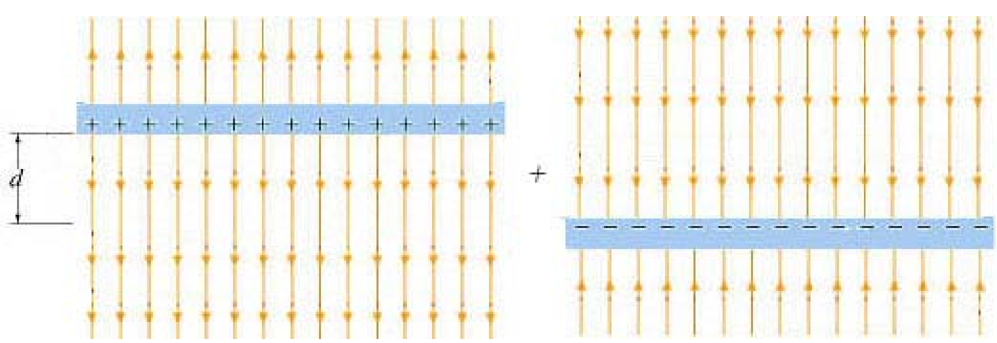

Figure 9.2 Electric field of positive and negative planes

Therefore, when we add these fields together, we see that the field outside the parallel planes is zero, and the field between the planes is twice the magnitude of the field of either plane.

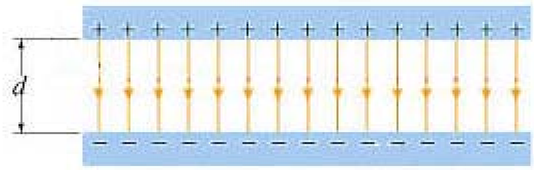

Figure 9.3 Electric field of parallel planes

Let's choose coordinates such that the planes are perpendicular to the x-axis, the positive plane is located at $x = d/2$, and the negative plane is located at $x = -d/2$. Then the electric field of the positive plane is given by (See Worked Examples: Coulomb Law),

$$\vec{\mathbf{E}}_{+} = \begin{cases} +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x > d/2 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x < d/2 \end{cases} \tag{9.3}$$

The electric field of the negative plane is given by

$$\vec{\mathbf{E}}_{-} = \begin{cases} -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x > -d/2 \\ +\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}}, & x < -d/2 \end{cases} \tag{9.4}$$

When we add these two fields together we get

$$\vec{\mathbf{E}} = \begin{cases} 0\,\hat{\mathbf{i}}, & x > d/2 \\ -\dfrac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}, & d/2 > x > -d/2 \\ 0\,\hat{\mathbf{i}}, & x < -d/2 \end{cases} \tag{9.5}$$

So the electric field between the plates is twice the magnitude of the field of a single plate

$$\boxed{\vec{\mathbf{E}} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}, \quad d/2 > x > -d/2} \tag{9.6}$$

Second Approach using Gauss's Law

We can also solve this problem using Gauss's Law. We still need the superposition principle to determine that the electric field is zero outside the plates. For the region $d/2 > x > -d/2$, we can now apply our methodology.

Step 1: The two surface charge distributions have a planar symmetry.

Step 2: We have already determined that the electric field is zero outside the planes and in uniform and points from the positive plane to the negative plane in between the planes.

Step 3: The surface charge densities divide space into three regions,

Let's define a x-coordinate axis with the positive non-conducting plane located at $x = +d/2$, and the negative plane at $x = -d/2$. The surface charge densities divide space into three regions,

7. $x > +d/2$
8. $-d/2 < x < +d/2$
9. $x < -d/2$

As before, we do not include the two non-conducting planes in our three regions.

Step 4: Although we have three regions of space, we need to choose our Gaussian surface so that

1. One end-cap of the Gaussian surface is in our region where the electric field is non-zero, $-d/2 < x < +d/2$.

2. Some of the surface charge is enclosed in the Gaussian surface.

Therefore we choose the Gaussian surface to be a cylindrical tube as shown in Figure 9.3 below. The tube has total length l and is evenly divided by the non-conducting plane. The tube has two end-caps, each of area A. End-cap 1 is located in the region where there is zero electric field, $x > +d/2$. End-cap 2 is located in the region $-d/2 < x < +d/2$.

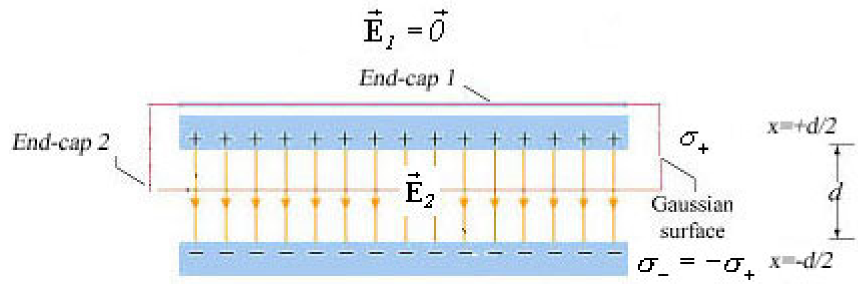

Figure 9.3 Gaussian surface for uniformly charged non-conducting plane

Step 5: As before, the Gaussian surface consists of three faces, S_1 (for end-cap 1), S_2 (for end-cap 2), and the open cylinder S_3. The total flux on S is the sum of the flux on the individual surfaces,

$$\Phi_E \equiv \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 + \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 + \iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 \tag{9.7}$$

On the open cylinder S_3 the electric field is tangent to the surface, so the flux is zero:

$$\iint_{S_3} \vec{\mathbf{E}}_3 \cdot d\vec{\mathbf{A}}_3 = 0 \tag{9.8}$$

On end-cap 1, the electric field is zero, $\vec{\mathbf{E}}_1 = \vec{0}$, so the flux in end-cap 1 is also zero:

$$\iint_{S_1} \vec{\mathbf{E}}_1 \cdot d\vec{\mathbf{A}}_1 = 0 \tag{9.9}$$

In region $-d/2 < x < +d/2$, the electric field is uniform and points out of end-cap 2, so the flux is

$$\iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 = EA \tag{9.10}$$

So the total flux on the closed Gaussian cylinder is $\Phi_E = \iint_{S_2} \vec{\mathbf{E}}_2 \cdot d\vec{\mathbf{A}}_2 = EA$.

Step 6: The area of the intersection of the positively charged plane with the Gaussian cylinder is equal to the area of the end-caps, A. Therefore the charge enclosed in the cylinder is

$$q_{in} = \sigma A \tag{9.11}$$

Step 7: We can now apply Gauss's Law and equate $\Phi_E = q_{in} / \varepsilon_0$.
This leads to

$$EA = \frac{\sigma A}{\varepsilon_0} \tag{9.12}$$

We can now solve for the magnitude of the electric field,

$$\boxed{E = \frac{\sigma}{\varepsilon_0}} \tag{9.13}$$

Step 8: We plot the x-component of the electric field as a function of the variable x.

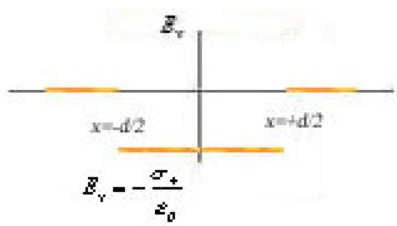

Figure 9.4 x-component of the electric field as a function x for two oppositely charges infinite planes

Although the magnitude of the electric field, E, is the same everywhere, we note that there are two discontinuity in the x-component of the electric field at $x = +d/2$ and $x = -d/2$. Recall that

$$\vec{\mathbf{E}} = \begin{cases} \vec{\mathbf{E}}_1 = \vec{0}, & x > +\frac{d}{2} \\ \vec{\mathbf{E}}_2 = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}, & +\frac{d}{2} > x > -\frac{d}{2} \\ \vec{\mathbf{E}}_3 = \vec{0}, & x < -\frac{d}{2} \end{cases} \tag{9.14}$$

At both surfaces, the electric field changes in magnitude by the amount

$$\begin{cases} \vec{\mathbf{E}}_1 - \vec{\mathbf{E}}_2 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}} & x = +\frac{d}{2} \\ \vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_3 = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}} & x = -\frac{d}{2} \end{cases} \tag{9.15}$$

Example 10: Electric Potential of a uniformly charged sphere of radius *a*

An insulated solid sphere of radius *a* has a uniform charge density ρ. Compute the electric potential everywhere.

Solution:

Using Gauss's law (see Example 4), the electric field due to the charge distribution is

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, & r > a \\ \dfrac{Qr}{4\pi\varepsilon_0 a^3}\hat{\mathbf{r}}, & r < a \end{cases}$$

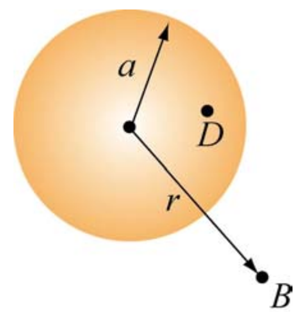

Figure 10.1

The electric potential at *B* (indicated in Figure 10.1) outside the sphere is

$$V_B(r) - V(\infty) = -\int_\infty^r \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \tag{10.1}$$

On the other hand, the electric potential at *D* inside the sphere is given by

$$\begin{aligned} V_D(r) - V(\infty) &= -\int_\infty^a dr E(r > a) - \int_a^r E(r < a) = -\int_\infty^a dr \frac{Q}{4\pi\varepsilon_0 r^2} - \int_a^r dr \frac{Qr}{4\pi\varepsilon_0 a^3} \\ &= \frac{1}{4\pi\varepsilon_0}\frac{Q}{a} - \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^3}\frac{1}{2}\left(r^2 - a^2\right) = \frac{1}{8\pi\varepsilon_0}\frac{Q}{a}\left(3 - \frac{r^2}{a^2}\right) \end{aligned} \tag{10.2}$$

A plot of electric potential as a function of *r* is given in Figure 10.2:

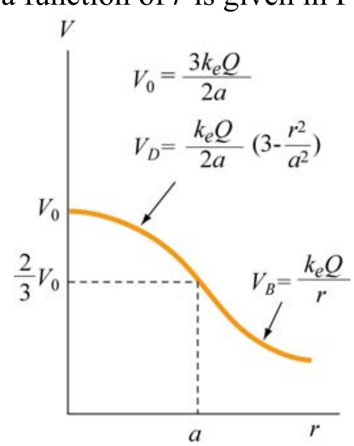

Figure 10.2 Electric potential due to a uniformly charged sphere as a function of *r*.