MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

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III. Electric Potential - Worked Examples

Example 1: Electric dipole

Example 2: Line of charge

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Example 1: Electric dipole

Consider an electric dipole along the *y*-axis, as shown in the Figure 1.1 below. We would like to derive the electric field at a point *P* on the *x*-*y* plane from the potential *V*.

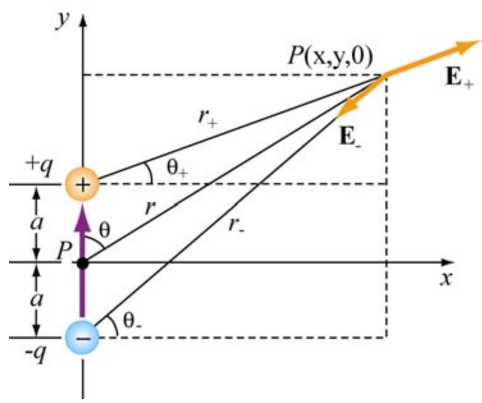

Figure 1.1 Electric dipole

By superposition principle, the potential at *P* is given by

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) \tag{1.1}$$

where $r_\pm^{\,2} = r^2 + a^2 \mp 2ra\cos\theta$. If we take the limit where $r \gg d$, then

$$\frac{1}{r_\pm} = \frac{1}{r}\left[1 + (a/r)^2 \mp 2(a/r)\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 \pm (a/r)\cos\theta + \cdots\right] \tag{1.2}$$

and the dipole potential can be approximated as

$$\begin{aligned} V &= \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta - 1 + \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right] \\ &\approx \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} \end{aligned} \tag{1.3}$$

$$\boxed{V = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}} \tag{1.4}$$

where $\vec{\mathbf{p}} = (2a)q\hat{\mathbf{j}}$ is the electric dipole moment. In spherical polar coordinates, the gradient operator is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\boldsymbol{\varphi}} \tag{1.5}$$

Since the potential is now a function of both r and θ, the electric field will have components along the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. Using Eq. (IV.D.5), we have

$$\boxed{E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^2}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^2}, \quad E_\phi = 0} \tag{1.6}$$

Example 2: Line of charge

Consider a non-conducting rod of length $2L$ having a uniform charge density λ. Find the electric potential at point P, a perpendicular distance y above the midpoint of the rod.

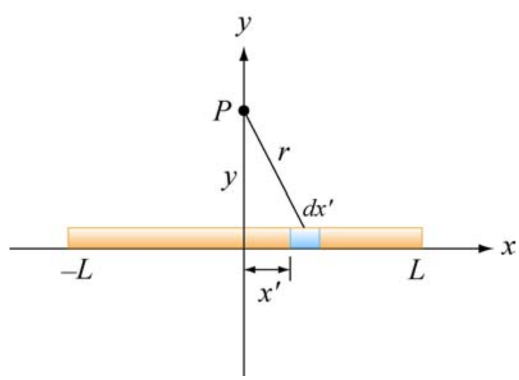

Figure 2.1 A non-conducting rod of length L and uniform charge density λ.

Solution:

Consider a differential element of length dx'. The charge carried by the element is $dq = \lambda dx'$. The source is located along the x-axis at the source point $(x',0)$ The field point is located on the y-axis at the point $(0,y)$. The distance from the source point to the field point is $r = \left(x'^2 + y^2\right)^{1/2}$. Its contribution to the potential at P is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx'}{(x'^2 + y^2)^{1/2}} \tag{2.1}$$

Taking V to be zero at infinity, the total potential due to the entire rod is

$$V=\frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dx'}{\sqrt{x'^2+y^2}}=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[x'+\sqrt{x'^2+y^2}\right]\Bigg|_{-L}^{L}=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{L+\sqrt{L^2+y^2}}{-L+\sqrt{L^2+y^2}}\right] \tag{2.2}$$

Where we have used the integration formula

$$\int\frac{dx'}{\sqrt{x'^2+y^2}}=\ln\left(x'+\sqrt{x'^2+y^2}\right) \tag{2.3}$$

In the limit $L \gg y,$ the potential becomes

$$\begin{aligned}V&=\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{L+L\sqrt{1+(y/L)^2}}{-L+L\sqrt{1+(y/L)^2}}\right]\approx\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{2L}{y^2/2L}\right)\\&=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{4L^2}{y^2}\right)=\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{L}{y}\right)\end{aligned} \tag{2.4}$$

The corresponding electric field can be obtained as

$$E_y=-\frac{\partial V}{\partial y}=\frac{\lambda}{2\pi\varepsilon_0 y} \tag{2.5}$$

in complete agreement with the result obtained before.

Example 3: Uniformly Charged Disk

Consider a uniformly charged disk of radius R and charge density σ. What is the electric potential at a distance x from the central axis?

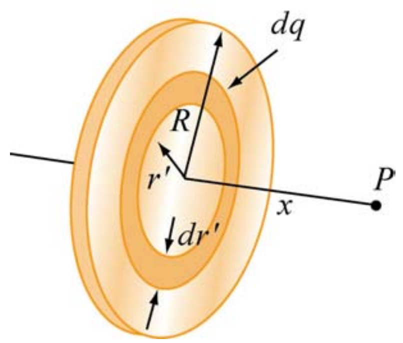

Figure 3.1 A non-conducting disk of radius R and uniform charge density σ.

Solution:

Consider a ring of radius r' and width dr'. The charge on the ring is given by

$dq' = \sigma dA' = \sigma(2\pi r' dr')$. The field point is located along the x-axis a distance x from the plane of the disk. The distance from the source to the field point is $r = \left(r'^2 + x^2\right)^{1/2}$. The potential at P due to the ring is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r' dr')}{\sqrt{r'^2 + x^2}} \tag{3.1}$$

By summing over all the rings that make up the disk, we have

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2 + x^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2 + x^2}\right]\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + x^2} - x\right] \tag{3.2}$$

In the limit $x >> R$, $\sqrt{R^2 + x^2} = x[1 + (R/x)^2]^{1/2} = x(1 + R^2/2x^2 + \cdots)$, and the potential simplifies to

$$V \approx \frac{\sigma}{2\varepsilon_0}\cdot\frac{R^2}{2x} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(\pi R^2)}{x} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{x} \tag{3.3}$$

which is the result for a point charge. In other words, at large distance, the potential due to a non-conducting charged disk is the same as that of a point charge Q. The electric field at P can easily be obtained as:

$$E_x = -\frac{\partial V}{\partial x} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{x}{\sqrt{R^2 + x^2}}\right] \tag{3.4}$$

In the limit $R \gg x$, $E_x \to \sigma/2\varepsilon_0$, which coincides with the result obtained for an infinite sheet.

Example 4: Change in Potential Energy for a Charge Moving Near a Uniformly Charged Wire

A thin rod extends along the z-axis from $z = -d$ to $z = d$. The rod carries a charge Q uniformly distributed along its length $2d$ with linear charge density $\lambda = Q/(2d)$. What is the change in potential energy when a particle of mass m and negative charge $q < 0$ moves from the point $z = 4d$ to the point $z = 3d$?

Solution:

From the Example 2 III.T.2 in the III.T Tutorial on Electric Potential, the electric potential difference between the point at infinity and a point on the z-axis with $z > d$, is

$$V(z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2d}\ln\left(\frac{z+d}{z-d}\right) \tag{4.1}$$

where we have chosen $V(\infty) = 0$. Therefore the electrical potential difference between the points infinity and $z = 4d$ is

$$V(z=4d) - V(\infty) = V(z=4d) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2d}\ln\left(\frac{4d+d}{4d-d}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2d}\ln\frac{5}{3} \tag{4.2}$$

Similarly, the electrical potential difference between the points infinity and $z = 3d$ is

$$V(z=3d) - V(\infty) = V(z=3d) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2d}\ln\left(\frac{3d+d}{3d-d}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2d}\ln 2 \tag{4.3}$$

We now use the fact that the electric potential difference between the points $z = 3d$ and $z = 4d$ is equivalent to taking the change in potential energy per test charge in moving a test charge from infinity to $z = 3d$ and then subtracting the change in potential energy per test charge in moving a test charge from infinity to $z = 4d$,

$$V(z=3d) - V(z=4d) = \left[V(z=3d) - V(\infty)\right] - \left[V(z=4d) - V(\infty)\right] \tag{4.4}$$

Therefore the electric potential difference between the points $z = 4d$ and $z = 3d$ is positive,

$$V(z=3d) - V(z=4d) = \frac{Q}{8\pi\varepsilon_0 d}\ln\frac{6}{5} > 0 \tag{4.5}$$

The change in potential energy when a particle of mass m and negative charge $q < 0$ moves from the point $z = 4d$ to the point $z = 3d$ is negative and given by

$$\Delta U = q\left[V(z=3d) - V(z=4d)\right] = q\frac{Q}{8\pi\varepsilon_0 d}\ln\frac{6}{5} < 0 \tag{4.6}$$

If the particle started out at rest at the point $z = 4d$ then the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^{\,2} \tag{4.7}$$

By conservation of energy, the change in kinetic energy is positive,

$$\Delta K = -\Delta U = -q\left[V(z=3d) - V(z=4\mathrm{d})\right] = -q\frac{Q}{8\pi\varepsilon_0 d}\ln\frac{6}{5} > 0 \tag{4.8}$$

So the magnitude of the velocity when the particle reaches the point $z = 3d$ is

$$v_f = \sqrt{\frac{-q}{4\pi\varepsilon_0}\frac{Q}{md}\ln\frac{6}{5}} \tag{4.9}$$

Example 5: Change in Kinetic Energy of a Charge Moving in a Uniform Electric Field

Consider a charge q moving between two parallel plates of equal and opposite uniform surface charge density σ, separated by a distance x (Figure 5.1). What is the magnitude of the velocity of the charge after it has been displaced from the initial position x_0 to the final position x_f ?

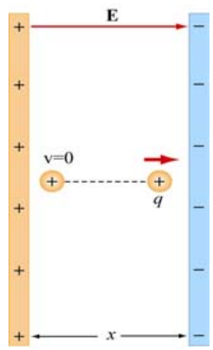

Figure 5.1

Solution:

From Example 16 in II.W Worked Examples of Coulomb's, if we neglect edge effects, we found that the electric field between the plates was uniform and equal to

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}\,. \tag{5.1}$$

From Example 3 in the Tutorial on Electric Potential, we found that the electric potential difference between the initial and final position is,

$$\Delta V = -\int_{x_0}^{x_f}\vec{\mathbf{E}}\cdot d\,\vec{\mathbf{r}} = -\int_{x_0}^{x_f}\frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}\cdot(dx\hat{\mathbf{i}}) = -\int_{x_0}^{x_f}\frac{\sigma}{\varepsilon_0}dx = -\frac{\sigma(x_f - x_0)}{\varepsilon_0} \tag{5.2}$$

Therefore the change in potential energy when the charge q moved from the initial position x_0 to the final position x_f is

$$\Delta U = q\Delta V = -\frac{q\sigma}{\varepsilon_0}(x_f - x_0) \tag{5.3}$$

Since the charge was released from rest, the change in kinetic energy is

$$\Delta K \equiv K_f - K_0 = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 \tag{5.4}$$

From conservation of energy,

$$\Delta K + \Delta U = \frac{1}{2}mv_f^2 + -\frac{q\sigma}{\varepsilon_0}(x_f - x_0) = 0 \tag{5.5}$$

So the magnitude of the velocity of the charge after it has been displaced from the initial position x_0 to the final position x_f is

$$v_f = \sqrt{\frac{2q\sigma}{m\varepsilon_0}(x_f - x_0)} \tag{5.6}$$