X. Faraday’s Law - Worked Examples

Example 1: Rectangular loop near a wire

An infinite straight wire carries a current \( I \) is placed above a rectangular loop of wire with width \( w \) and length \( L \), as shown in the figure below.

(a) Determine the magnetic flux through the rectangular loop due to the current \( I \).

(b) Suppose that the current is a function of time with \( I(t) = a + b t \), where \( a \) and \( b \) are positive constants. What is the induced emf in the loop and the direction of the induced current?

Solution:

(a) Using Ampere’s law:

\[
\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{\text{enc}} \tag{1.1}
\]

the magnetic field due to a current-carrying wire at a distance \( r \) away is

\[
B = \frac{\mu_0 I}{2\pi r} \tag{1.2}
\]
The total magnetic flux $\Phi_B$ through the loop can be obtained by summing over contributions from all differential area elements $dA = L \, dr$:

$$\Phi_B = \int d\Phi_B = \int \mathbf{B} \cdot d\mathbf{A} = \frac{\mu_0 LL}{2\pi} \int_{h}^{h+w} \frac{dr}{r} = \frac{\mu_0 LL}{2\pi} \ln \left( \frac{h+w}{h} \right) \quad (1.3)$$

(b) According to Faraday’s law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \left[ \frac{\mu_0 LL}{2\pi} \ln \left( \frac{h+w}{h} \right) \right] = -\frac{\mu_0 LL}{2\pi} \ln \left( \frac{h+w}{h} \right) \frac{dI}{dt} \quad (1.4)$$

where we have used $\frac{dI}{dt} = b$.

The straight wire carrying a current $I$ produces a magnetic flux into the page through the rectangular loop. By Lenz’s law, the induced current in the loop must be flowing **counterclockwise** in order to produce a magnetic field out of the page to counteract the increase in inward flux.
Example 2: Changing area in a square loop

A square loop with length $l$ on each side is placed in a uniform magnetic field pointing into the page. During a time interval $t$, the loop is pulled from its two edges and turned into a rhombus, as shown in the figure below. Assuming that the total resistance of the loop is $R$, find the induced current in the loop and its direction.

![Diagram of square to rhombus loop transition](image)

Solution:

Using Faraday’s law, we have

$$
\varepsilon = -\frac{\Delta \Phi}{\Delta t} = -B \left(\frac{\Delta A}{\Delta t}\right)
$$

(2.1)

Since the initial and the final areas of the loop are $A_i = l^2$ and $A_f = l^2 \sin \theta$, respectively (recall that the area of a parallelogram defined by two vectors $\vec{l}_1$ and $\vec{l}_2$ is $A = |\vec{l}_1 \times \vec{l}_2| = l_1 l_2 \sin \theta$), the rate of change of area is

$$
\frac{\Delta A}{\Delta t} = \frac{A_f - A_i}{t} = -\frac{l^2 (1 - \sin \theta)}{t} < 0
$$

(2.2)

which gives

$$
\varepsilon = \frac{Bl^2 (1 - \sin \theta)}{t} > 0
$$

(2.3)

Thus, the induced current is

$$
I = \frac{\varepsilon}{R} = \frac{Bl^2 (1 - \sin \theta)}{tR}
$$

(2.4)

Since $\Delta A / \Delta t < 0$, the magnetic flux into the page decreases. Hence, the current flows in the clockwise direction to compensate the change.
Example 3: Sliding rod

A conducting rod of length $l$ is free to slide on two parallel conducting bars as shown below.

In addition, two resistors $R_1$ and $R_2$ are connected across the ends of the bars. There is a uniform magnetic field pointing into the page. Suppose an external agent pulls the bar to the left at a constant speed $v$. Find the following:

(a) the currents through both resistors;
(b) the total power delivered to the resistors;
(c) the applied force needed for the rod to maintain a constant velocity.

Solution:

(a) The emf induced between the ends of the moving rod is

$$

\varepsilon = -\frac{d\Phi_B}{dt} = -Blv

$$

(3.1)

The currents through the resistors are

$$

I_1 = \frac{\varepsilon}{R_1}, \quad I_2 = \frac{\varepsilon}{R_2}

$$

(3.2)

Since the flux into the page for the left loop is decreasing, $I_1$ flows clockwise to produce a magnetic field pointing into the page. On the other hand, the flux into the page for the right loop is increasing. To compensate the change, according to Lenz’s law, $I_2$ must flow counterclockwise to produce a magnetic field pointing out of the page.

(b) The total power dissipated in the two resistors is

$$

P_r = I_1 |\varepsilon| + I_2 |\varepsilon| = (I_1 + I_2) |\varepsilon| = \varepsilon^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = B^2 l^2 v^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)

$$

(3.3)
(c) The total current flowing through the rod is \( I = I_1 + I_2 \). Thus, the magnetic force acting on the it is

\[
F_B = I_1 B = I_2 B \left( \frac{1}{R_1} + \frac{1}{R_2} \right) = B^2 I_1^2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)
\]

and the direction is to the right. Thus, an external agent must apply an equal but opposite force \( \vec{F}_{ext} = -\vec{F}_B \) to the left in order to maintain a constant speed.

Alternatively, we note that since the power dissipated through the resistors must be equal to \( P_{ext} \), the mechanical power supplied by the external agent. Since

\[
P_{ext} = \vec{F}_{ext} \cdot \vec{v} = F_{ext} v
\]

the same result is obtained.
Example 4: Moving bar

A conducting rod of length $l$ moves with a constant velocity $v$ perpendicular to an infinitely long, straight wire carrying a current $I$, as shown in the figure below.

What is the emf generated between the ends of the rod?

Solution:

From Faraday’s law, the motional emf is

$$|\varepsilon| = Blv$$

(4.1)

where $v$ is the speed of the rod. However, the magnetic field due to the straight current-carrying wire at a distance $r$ away is, using Ampere’s law:

$$B = \frac{\mu_0 I}{2\pi r}$$

(4.2)

Thus, the emf between the ends of the rod is given by

$$|\varepsilon| = \left(\frac{\mu_0 I}{2\pi r}\right)lv$$

(4.3)
Example 5: Time-varying B field

A circular loop of wire of radius $a$ is placed in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field. The magnetic field varies with time according to $B(t) = B_0 + bt$, where $a$ and $b$ are constants.

(a) Calculate the magnetic flux through the loop at $t = 0$.

(b) Calculate the induced emf in the loop.

(c) What is the induced current if the overall resistance of the loop is $R$?

(d) Find the power dissipated due to the resistance of the loop?

Solution:

(a) The magnetic flux at time $t$ is given by

$$\Phi_B = BA = (B_0 + bt)(\pi a^2) = \pi(B_0 + bt)a^2$$

Therefore, at $t = 0$,

$$\Phi_B = \pi B_0 a^2$$

(b) Using Faraday’s Law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -A \frac{dB}{dt} = -(\pi a^2) \frac{d}{dt}(B_0 + bt) = -\pi ba^2$$

(c) The induced current is

$$I = \frac{\varepsilon}{R} = \frac{\pi ba^2}{R}$$

(d) The power dissipated due to the resistance $R$ is

$$P = I^2 R = \left(\frac{\pi ba^2}{R}\right)^2 R = \frac{(\pi ba^2)^2}{R}$$
Example 6: Moving loop

A rectangular loop of dimensions $l$ and $w$ moves with a constant velocity $v$ away from an infinitely long straight wire carrying a current $I$ in the plane of the loop. Let the total resistance of the loop be $R$. What is the current in the loop at the instant the near side is a distance $r$ from the wire?

Solution:

The magnetic field at a distance $s$ from the straight wire is, using Ampere’s law:

$$B = \frac{\mu_0 I}{2\pi s}$$  \hspace{1cm} (6.1)

The magnetic flux through a differential area element $dA=lds$ of the loop is

$$d\Phi_B = \vec{B} \cdot d\vec{A} = \frac{\mu_0 I}{2\pi s} l ds$$  \hspace{1cm} (6.2)

Integrating over the entire area of the loop, the total flux is

$$\Phi_B = \frac{\mu_0 I}{2\pi} \int_r^{r+w} \frac{ds}{s} = \frac{\mu_0 I}{2\pi} \ln \left( \frac{r+w}{r} \right)$$  \hspace{1cm} (6.3)

Differentiating with respect to $t$, we obtain the induced emf as

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 Il}{2\pi} \left( \ln \frac{r+w}{r} \right) = -\frac{\mu_0 Il}{2\pi} \left( \frac{1}{r+w} - \frac{1}{r} \right) \frac{dr}{dt} = \frac{\mu_0 Il w}{2\pi} \frac{v}{r(r+w)}$$  \hspace{1cm} (6.4)

where $v = \frac{dr}{dt}$. The induced current is

$$I = \frac{|\varepsilon|}{R} = \frac{\mu_0 Il v}{2\pi R r (r+w)}$$  \hspace{1cm} (6.5)