Example 1: Plane Electromagnetic Wave

Suppose the electric field of a plane electromagnetic wave is given by

\[ \mathbf{E}(z,t) = E_0 \cos(kz - \omega t) \hat{i} \]  

Determine

(a) the direction of wave propagation.

(b) the corresponding magnetic field \( \mathbf{B} \).

Solution:

(a) By writing the argument of the cosine function as \( kz - \omega t = k(z - ct) \) with \( \omega = ck \), we see that the wave is traveling in the +z direction.

(b) The direction of propagation of the electromagnetic waves coincides with the direction of the Poynting vector which is given by \( \mathbf{S} = \mathbf{E} \times \mathbf{B} \). In addition, \( \mathbf{E} \) and \( \mathbf{B} \) are perpendicular to each other. Therefore, if \( \mathbf{E} = E(z,t) \hat{i} \) and \( \mathbf{S} = \mathbf{S} \hat{k} \), then \( \mathbf{B} = B(z,t) \hat{j} \), i.e., \( \mathbf{B} \) points in the +y direction. Since \( \mathbf{E} \) and \( \mathbf{B} \) are in phase with each other, one may write

\[ \mathbf{B}(z,t) = B_0 \cos(kz - \omega t) \hat{j} \]  

To find the magnitude of \( \mathbf{B} \), we make use of Faraday’s law:

\[ \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \]  

which implies

\[ \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \]  

From the above equations, we obtain

\[ -E_0 k \sin(kz - \omega t) = -B_0 \omega \sin(kz - \omega t) \]  

or

\[ 1 \]
Thus, the magnetic field is given by

$$\mathbf{B}(z,t) = (E_0 / c) \cos(kz - \omega t) \hat{j}$$

(3.1)

**Example 2: Wave equation**

Verify that

$$E(x,t) = E_0 \cos(kx - \omega t)$$

(2.1)

and

$$B(x,t) = B_0 \cos(kx - \omega t)$$

(2.2)

satisfy the one-dimensional wave equation:

$$\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \begin{bmatrix} E(x,t) \\ B(x,t) \end{bmatrix} = 0$$

(2.3)

**Solution:**

Differentiating $E = E_0 \cos(kx - \omega t)$ with respect to $x$, we get

$$\frac{\partial E}{\partial x} = -kE_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial x^2} = -k^2 E_0 \cos(kx - \omega t)$$

(2.4)

Similarly, differentiating with respect to $t$ yields

$$\frac{\partial E}{\partial t} = \omega E_0 \sin(kx - \omega t), \quad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E_0 \cos(kx - \omega t)$$

(2.5)

Thus,

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \left( -k^2 + \frac{\omega^2}{c^2} \right) E_0 \cos(kx - \omega t) = 0$$

(2.6)

where we have made used of the equation $\omega = kc$.

One may follow a similar procedure to prove the magnetic case.
**Example 3: Transverse Traveling Plane Waves**

Show that the components of the electromagnetic plane wave are zero in the direction of propagation.

**Solution:**

Let the components of the plane electromagnetic wave be traveling in the positive $x$-direction be

$$
\vec{E}(x,t) = E_x(x,t)\hat{j} = E_0 \sin(kx - \omega t)\hat{j} \tag{3.1}
$$

$$
\vec{B}(x,t) = B_z(x,t)\hat{k} = B_0 \sin(kx - \omega t)\hat{k} \tag{3.2}
$$

with $B_0 = E_0 / c$. The electric field component in the $x$-direction is either constant or zero. This follows from Gauss’s Law. To see this let’s suppose that there is only a non-zero component of the electric field of a plane electromagnetic wave in the $x$-direction,

$$
\vec{E}(x,t) = E_x(x,t)\hat{i} \tag{3.3}
$$

Choose a Gaussian box with the sides oriented along the axes such that one side lies on the plane given by $x = x_0$ and another side lies on the plane given by $x = x_1$. Both sides have area $A$.

Since the flux of the electric field through the box is zero (no charge enclosed), we have

$$
\oint_S \vec{E} \cdot d\vec{A} = E_x(x_1)A - E_x(x_0)A = 0 \tag{3.4}
$$

the electric field must satisfy $E_x(x_1) = E_x(x_0)$ for all values of $x_0$ and $x_1$. Therefore the electric field must be constant. However, we must exclude constant fields because they store infinite energy.

Similarly, the magnetic field also is a function of only $x$ and $t$ and must be perpendicular to the $x$-direction. The argument follows from Gauss’s law for magnetism:

$$
\oint_S \vec{B} \cdot d\vec{A} = 0 \tag{3.5}
$$

Since our field vectors only have components that are perpendicular to the direction of propagation, this type of traveling wave is known as a **transverse traveling wave**.
Example 4: Poynting vector

A parallel-plate capacitor with circular plates of radius $R$ and separated by a distance $d$ is charged through a straight wire carrying current $I$, as shown in the figure below:

(a) Show that as the capacitor is being charged, the Poynting vector $\mathbf{S}$ points radially inward toward the center of the capacitor.

(b) By integrate $\mathbf{S}$ over the cylindrical boundary, show that the rate at which energy enters the capacitor is equal to the rate at which electrostatic energy is being stored in the electric field.

Solution:

(a) Let the axis of the circular plates be the $z$-axis. Suppose at some instant the amount of charge accumulated on the positive plate is $+Q$. The electric field is

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0} \mathbf{k} = \frac{Q}{\pi R^2 \varepsilon_0} \mathbf{k}$$

(4.1)

According to Ampere-Maxwell’s equation, a magnetic field can be induced by changing electric flux:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d}{dt} \oint \mathbf{E} \cdot d\mathbf{A}$$

From the cylindrical symmetry of the system, we see that the magnetic field will be circular, centered on the $z$-axis, i.e., $\mathbf{B} = B \mathbf{\phi}$

Consider a circular path of radius $r < R$. Using the above formula, we obtain
\[ B(2\pi r) = 0 + \mu_0 \varepsilon_0 \frac{d}{dt} \left( \frac{Q}{\pi R^2 \varepsilon_0} \pi r^2 \right) = \frac{\mu_0 r^2}{R^2} \frac{dQ}{dt} \]  

(4.2)

or

\[ \vec{B} = \frac{\mu_0 r}{2\pi R^2} \frac{dQ}{dt} \hat{\phi} \]  

(4.3)

The Poynting \( \vec{S} \) vector can then be written as

\[ \vec{S} = \frac{1}{\mu_0} \hat{E} \times \vec{B} = \frac{1}{\mu_0} \left( \frac{Q}{\pi R^2 \varepsilon_0} \hat{\mathbf{k}} \right) \times \left( \frac{\mu_0 r}{2\pi R^2} \frac{dQ}{dt} \hat{\phi} \right) = - \left( \frac{Q r}{2\pi^2 R^4 \varepsilon_0} \right) \left( \frac{dQ}{dt} \right) \hat{r} \]  

(4.4)

Note that it points in the \(-\hat{r}\) direction, or radially inward toward the center of the capacitor.

(b) The energy per unit volume carried by the electric field is \( u_E = \varepsilon_0 E^2 / 2 \). The total energy stored in the electric field then becomes

\[ U_E = u_E V = \frac{\varepsilon_0}{2} E^2 \left( \pi R^2 d \right) = \frac{1}{2} \varepsilon_0 \left( \frac{Q}{\pi R^2 \varepsilon_0} \right)^2 \pi R^2 d = \frac{dQ^2}{2\pi R^2 \varepsilon_0} \]  

(4.5)

Differentiating the above expression with respect to \( t \), we obtain the rate at which this energy is being stored:

\[ \frac{dU_E}{dt} = \frac{d}{dt} \left( \frac{dQ^2}{2\pi R^2 \varepsilon_0} \right) = \frac{Q d}{\pi R^2 \varepsilon_0} \frac{dQ}{dt} \]  

(4.6)

On the other hand, the rate at which energy flows into the capacitor through the cylinder at \( r = R \) can be obtained by integrating \( \vec{S} \) over the surface area:

\[ \oint \vec{S} \cdot d\vec{A} = S A_R = \left( \frac{Q r}{2\pi^2 \varepsilon_0 R^4} \frac{dQ}{dt} \right) (2\pi R d) = \frac{Q d}{\varepsilon_0 \pi R^2} \frac{dQ}{dt} \]  

(4.7)

which is equal to the rate at which energy is being stored in the electric field.
Example 5: Antenna

Suppose an AM radio station broadcasts isotropically with an average power of \(< S >\). A dipole antenna with length \(l\) is receiving the signal at a distance \(d\) from the transmitter. What is the emf between the ends of the antenna?

**Solution:**

The intensity of the electromagnetic wave is given by

\[
I = < S >= \frac{< P >}{A} = \frac{< P >}{\pi d^2}
\]  

(4.1)

Since the time-averaged value of \(S\) is

\[
< S >= \left\langle \frac{E^2}{\mu_0} \right\rangle = \frac{E_0^2}{2\mu_0 c}
\]  

(4.2)

we have

\[
E_0 = \sqrt{2\mu_0 c < S >} = \sqrt{\frac{\mu_0 c < S >}{2\pi d^2}}
\]  

(4.3)

The emf between the ends of the antenna then becomes

\[
\varepsilon = E_0 l = l \sqrt{2\mu_0 c < S >} = l \sqrt{\frac{\mu_0 c < S >}{2\pi d^2}}
\]  

(4.4)