W13D2: Displacement Current, Maxwell’s Equations, Wave Equations

Today’s Reading Course Notes: Sections 13.1-13.4
Announcements

Math Review Tuesday May 6 from 9 pm-11 pm in 26-152

Pset 10 due May 6 at 9 pm
Outline

Displacement Current

Poynting Vector and Energy Flow

Maxwell’s Equations
Maxwell’s Equations

\[
\oint \mathbf{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV \quad \text{(Gauss's Law)}
\]

\[
\oint \mathbf{B} \cdot \hat{n} \, da = 0 \quad \text{(Magnetic Gauss's Law)}
\]

\[
\oint_{c} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_s \mathbf{B} \cdot \hat{n} \, da \quad \text{(Faraday's Law)}
\]

\[
\oint_{c} \mathbf{B} \cdot d\mathbf{s} = \mu_0 \iiint_S \mathbf{J} \cdot \hat{n} \, da \quad \text{(Ampere's Law quasi - static)}
\]

Is there something missing?
Maxwell’s Equations
One Last Modification: Displacement Current
Ampere’s Law: Capacitor

Consider a charging capacitor:

Use Ampere’s Law to calculate the magnetic field just above the top plate.

\[
\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I_{enc}
\]

\[
I_{enc} = \int \int \mathbf{J} \cdot \mathbf{n} \, da
\]

1) Surface \( S_1 \): \( I_{enc} = I \)

2) Surface \( S_2 \): \( I_{enc} = 0 \)

What’s Going On?
Displacement Current

We don’t have current between the capacitor plates but we do have a changing E field. Can we “make” a current out of that?

\[ E = \frac{Q}{\varepsilon_0 A} \Rightarrow Q = \varepsilon_0 EA = \varepsilon_0 \Phi_E \]

\[ \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt} \equiv I_{\text{dis}} \]

This is called the “displacement current”. It is not a flow of charge but proportional to changing electric flux.
Displacement Current:

\[ I_{\text{dis}} = \varepsilon_0 \frac{d}{dt} \int \int_S \mathbf{E} \cdot \mathbf{n} \, da = \varepsilon_0 \frac{d\Phi_E}{dt} \]

If surface \(S_2\) encloses all of the electric flux due to the charged plate then \(I_{\text{dis}} = I\)
Maxwell-Ampere’s Law

\[ \oint_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_{0} \iint_{S} \mathbf{J} \cdot \mathbf{n} \, da + \mu_{0} \varepsilon_{0} \frac{d}{dt} \iint_{S} \mathbf{E} \cdot \mathbf{n} \, da \]

\[ = \mu_{0} (I_{\text{enc}} + I_{\text{dis}}) \]

“flow of electric charge”

\[ I_{\text{enc}} = \iint_{S} \mathbf{J} \cdot \mathbf{n} \, da \]

“changing electric flux”

\[ I_{\text{dis}} = \varepsilon_{0} \frac{d}{dt} \iint_{S} \mathbf{E} \cdot \mathbf{n} \, da \]
Concept Question: Capacitor

If instead of integrating the magnetic field around the pictured Amperian circular loop of radius \( r \) we were to integrate around an Amperian loop of the same radius \( R \) as the plates (b) then the integral of the magnetic field around the closed path would be

1. the same.
2. larger.
3. smaller.
Sign Conventions: Right Hand Rule

\[ \oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \iint \mathbf{J} \cdot \hat{n} \, da + \mu_0 \epsilon_0 \frac{d}{dt} \iint \mathbf{E} \cdot \hat{n} \, da \]

Integration direction clockwise for line integral requires that unit normal points into page for surface integral.

Current positive into the page. Negative out of page.

Electric flux positive into page, negative out of page.
Sign Conventions: Right Hand Rule

\[ \oint_{c} \vec{B} \cdot d\vec{s} = \mu_{0} \iint_{s} \vec{J} \cdot \hat{n} \, da + \mu_{0} \varepsilon_{0} \frac{d}{dt} \iint_{s} \vec{E} \cdot \hat{n} \, da \]

Integration direction counter clockwise for line integral requires that unit normal points out page for surface integral.

Current positive out of page. Negative into page.

Electric flux positive out of page, negative into page.
Concept Question: Capacitor

Consider a circular capacitor, with an Amperian circular loop (radius $r$) in the plane midway between the plates. When the capacitor is charging, the line integral of the magnetic field around the circle (in direction shown) is

1. Zero (No current through loop)
2. Positive
3. Negative
4. Can’t tell (need to know direction of E)
Concept Question: Capacitor

The figures above shows a side and top view of a capacitor with charge $Q$ and electric and magnetic fields $E$ and $B$ at time $t$. At this time the charge $Q$ is:

1. Increasing in time
2. Constant in time.
3. Decreasing in time.
A circular capacitor of spacing \( d \) and radius \( R \) is in a circuit carrying the steady current \( i \) shown. At time \( t = 0 \), the plates are uncharged.

1. Find the electric field \( E(t) \) at \( P \) vs. time \( t \) (mag. & dir.)
2. Find the magnetic field \( B(t) \) at \( P \)
Energy Flow
Poynting Vector

Power per unit area: Poynting vector

\[ \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \]

Power through a surface

\[ P = \int \int \vec{S} \cdot \hat{n} \, da \]

(open surface)
What is the magnetic field?
The figures above show a side and top view of a capacitor with charge $Q$ and electric and magnetic fields $E$ and $B$ at time $t$. At this time the energy stored in the electric field is:

1. Increasing in time.
2. Constant in time.
3. Decreasing in time.
Energy Flow: Capacitor

\[ \tilde{S} = (\vec{E} \times \vec{B}) / \mu_0 = (E \hat{k} \times B \hat{\theta}) / \mu_0 = -(EB / \mu_0) \hat{r} \]

\[ P = \iiint_{\text{cylindrical shell}} \tilde{S} \cdot \hat{n} \text{out} \, da = \iint_{\text{cylindrical body}} -(EB / \mu_0) \hat{r} \cdot \hat{r} \, da = -(EB / \mu_0) 2\pi Rh \]

\[ P = -\left( \frac{EB}{\mu_0} \right) 2\pi Rh \]

\[ = -\frac{E}{\mu_0} \left( \frac{\mu_0 \varepsilon_0}{2} \frac{dE}{dt} R \right) 2\pi Rh \]

\[ = -\varepsilon_0 E \frac{dE}{dt} \pi R^2 h \]

\[ = -\frac{d}{dt} \left( \frac{1}{2} \varepsilon_0 E^2 \right) \pi R^2 h \]

\[ = -\left( \frac{dU_E}{dt} \right) (Volume) \]

\[ B2\pi R = \mu_0 \varepsilon_0 \frac{dE}{dt} \pi R^2 \Rightarrow B = \frac{\mu_0 \varepsilon_0}{2} \frac{dE}{dt} R \]
Maxwell’s Equations

\[ \int_S \vec{E} \cdot \hat{n} \, da = \frac{1}{\varepsilon_0} \int_V \rho \, dV \]  
(Gauss's Law)

\[ \int_S \vec{B} \cdot \hat{n} \, da = 0 \]  
(Magnetic Gauss's Law)

\[ \oint_C \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} \, da \]  
(Faraday's Law)

\[ \oint_C \vec{B} \cdot d\vec{s} = \mu_0 \int_S \vec{J} \cdot \hat{n} \, da + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n} \, da \]  
(Maxwell - Ampere's Law)
Electromagnetism Review

E fields are associated with:
(1) electric charges (Gauss’s Law )
(2) time changing B fields (Faraday’s Law)

B fields are associated with
(3a) moving electric charges (Ampere-Maxwell Law)
(3b) time changing E fields (Maxwell’s Addition (Ampere-Maxwell Law)

Conservation of magnetic flux
(4) No magnetic charge (Gauss’s Law for Magnetism)
Electromagnetism Review

Conservation of charge:

\[ \oint \mathbf{J} \cdot d\mathbf{A} = -\frac{d}{dt} \iiint \rho \, dV \]

closed

surface

volume

enclosed

E and B fields exert forces on (moving) electric charges:

\[ \mathbf{F}_q = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

Energy stored in electric and magnetic fields

\[ U_E = \iiint u_E \, dV = \iiint \frac{\mathcal{E}_0}{2} E^2 \, dV \]

all space

all space

\[ U_B = \iiint u_B \, dV = \iiint \frac{1}{2\mu_0} B^2 \, dV \]

all space

all space
Maxwell’s Equations in Vacua
Maxwell’s Equations

1. $\iiint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\varepsilon_0}$ (Gauss's Law)

2. $\iiint_S \vec{B} \cdot d\vec{A} = 0$ (Magnetic Gauss's Law)

3. $\oint_C \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$ (Faraday's Law)

4. $\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$ (Ampere - Maxwell Law)

What about free space (no charge or current)?
James Clerk Maxwell is sometimes credited as being the father of additive color. He had the photographer Thomas Sutton photograph a tartan ribbon on black-and-white film three times, first with a red, then green, then blue color filter over the lens. The three black-and-white images were developed and then projected onto a screen with three different projectors, each equipped with the corresponding red, green, or blue color filter used to take its image. When brought into alignment, the three images (a black-and-red image, a black-and-green image and a black-and-blue image) formed a full color image, thus demonstrating the principles of additive color.

http://upload.wikimedia.org/wikipedia/commons/7/7f/Tartan_Ribbon.jpg
How Do Maxwell’s Equations Lead to EM Waves?
Wave Equation

Start with Ampere-Maxwell Eq and closed oriented loop

\[ \oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot \mathbf{n} \, da \]
Wave Equation

Start with Ampere-Maxwell Eq:

Apply it to red rectangle:

\[ \oint_c \mathbf{B} \cdot d\mathbf{s} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot \hat{n} \, da \]

\[ \mu_0 \varepsilon_0 \frac{d}{dt} \int \mathbf{E} \cdot \hat{n} \, da \]

\[ = \mu_0 \varepsilon_0 \left( l \Delta x \frac{\partial E_y (x + \Delta x / 2, t)}{\partial t} \right) \]

\[ - \frac{B_z (x + \Delta x, t) - B_z (x, t)}{\Delta x} = \mu_0 \varepsilon_0 \frac{\partial E_y (x + \Delta x / 2, t)}{\partial t} \]

So in the limit that \( dx \) is very small:

\[ - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]
Group Problem: Wave Equation

Use Faraday’s Law
\[ \oint_{C} \bar{E} \cdot d\bar{s} = -\frac{d}{dt} \int \bar{B} \cdot \hat{n} da \]
and apply it to red rectangle to find the partial differential equation in order to find a relationship between

\[ \frac{\partial E_y}{\partial x} \text{ and } \frac{\partial B_z}{\partial t} \]
Group Problem: Wave Equation Sol.

Use Faraday’s Law:

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n} \, da \]

and apply it to red rectangle:

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{s} = E_{y}(x + \Delta x, t)l - E_{y}(x, t)l \]

\[ -\frac{d}{dt} \int \mathbf{B} \cdot \mathbf{n} \, da = -ldx \frac{\partial B_{z}}{\partial t} \]

\[ \frac{E_{y}(x + dx, t) - E_{y}(x, t)}{dx} = -\frac{\partial B_{z}}{\partial t} \]

So in the limit that \( dx \) is very small:

\[ \frac{\partial E_{y}}{\partial x} = -\frac{\partial B_{z}}{\partial t} \]
1D Wave Equation for Electric Field

\[
\frac{\partial E_y}{\partial x} = - \frac{\partial B_z}{\partial t} \quad (1) \\
- \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \quad (2)
\]

Take x-derivative of Eq.(1) and use the Eq. (2)

\[
\frac{\partial^2 E_y}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial E_y}{\partial x} \right) = \frac{\partial}{\partial x} \left( - \frac{\partial B_z}{\partial t} \right) = - \frac{\partial}{\partial t} \left( \frac{\partial B_z}{\partial x} \right) = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]

\[
\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}
\]
1D Wave Equation for E

\[ \frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \]

This is an equation for a wave. Let \( E_y = f(x - vt) \)

\[ \begin{align*}
\frac{\partial^2 E_y}{\partial x^2} &= f''(x - vt) \\
\frac{\partial^2 E_y}{\partial t^2} &= v^2 f''(x - vt)
\end{align*} \]

\[ \Rightarrow v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]
Definition of Constants and Wave Speed

Recall exact definitions of

\[ c \equiv 299792458 \text{ m} \cdot \text{s}^{-1} \]
\[ \mu_0 \equiv 4\pi \times 10^{-7} \text{ N} \cdot \text{s}^2 \cdot \text{C}^{-2} \]

The permittivity of free space \( \varepsilon_0 \) is exactly defined by

\[ \varepsilon_0 \equiv \frac{1}{c^2 \mu_0} \equiv 8.854187817 \times 10^{-12} \text{ C}^2 \cdot \text{m}^{-2} \cdot \text{N}^{-1} \]

\[ \Rightarrow v = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \text{ in vacua} \]
Group Problem: 1D Wave Eq. for B

\[
\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \quad \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}
\]

Take appropriate derivatives of the above equations and show that

\[
\frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B_z}{\partial t^2}
\]
Wave Equations: Summary

Both electric & magnetic fields travel like waves:

\[ \frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 E_y}{\partial t^2} \]
\[ \frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 B_z}{\partial t^2} \]

with speed

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \]

But there are strict relations between them:

\[ \frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \]
\[ \frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \]
Electromagnetic Waves
Electromagnetic Radiation: Plane Waves

http://youtu.be/3lvZF_LXzcc