Problem Solving 5b: Magnetic Fields & Biot-Savart

**Biot-Savart Law:**

Moving charges (currents) are sources for magnetic fields. When the moving charges form a current $I$ in a wire, the magnetic field at any point $P$ due to the current can be calculated by adding up the magnetic field contributions, $d\mathbf{B}$, from small segments of the wire $ds$, (see figure below).

![Diagram of magnetic field and current](image)

The small segment can be thought of as a vector quantity with magnitude equal to the length of the segment and the direction of $ds$ is given by the direction of the tangent to the segment pointing in the direction of the current. The infinitesimal current source is then $Id\mathbf{s}$. Let $r$ denote the distance from the current source to the field point $P$. Let $\hat{\mathbf{r}}$ be a unit vector that points in the direction from the current source to the field point.

The Biot-Savart Law gives an expression for the magnetic field contributions, $d\mathbf{B}$, from the current source, $Id\mathbf{s}$,

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{s} \times \hat{\mathbf{r}}}{r^2}$$

where $\mu_0$ is a constant called the **permeability of free space**:

$$\mu_0 = 4\pi \times 10^{-7} \, \text{T} \cdot \text{m} \cdot \text{A}^{-1}.$$  

Adding up these contributions to find the magnetic field at the point $P$ requires integrating over the current source,

$$\mathbf{B}(P) = \int_{\text{wire}} d\mathbf{B} = \int_{\text{wire}} \frac{\mu_0}{4\pi} \frac{Id\mathbf{s} \times \hat{\mathbf{r}}}{r^2}.$$
The integral is a vector integral. That means that the expression for \( \mathbf{B} \) is really three integrals, one for each component of \( \mathbf{B} \). The vector nature of this integral appears in the cross product \( l d \hat{s} \times \hat{r} \). Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart Law.

When we introduce a coordinate system with origin we can rewrite the unit vector \( \hat{r} \) from the current element vector \( l d \hat{s} \) to the field point \( P \) as follows. Define the vector \( \hat{r}' \) to be the vector from the origin to the current element \( l d \hat{s} \). Define the vector \( \hat{r} \) to be the vector from the origin to the field point \( P \). Then the distance from the current element to the field point is \( r = |\hat{r} - \hat{r}'| \) and the unit vector \( \hat{r} \) is

\[
\hat{r} = \frac{\hat{r} - \hat{r}'}{|\hat{r} - \hat{r}'|}.
\]

The Biot–Savart Law can then be rewritten as

\[
\mathbf{B}(\hat{r}) = \frac{\mu_0}{4\pi} \int_{wire} \frac{l d \hat{s}' \times (\hat{r} - \hat{r}')}{|\hat{r} - \hat{r}'|^3}.
\]
Problem 1: Straight Wire

Consider a straight wire of length $L$ that has a current $I$ flowing in the wire. (We will not worry about the return path of the current or the source for the current.) In this problem we will try to set up an integral vector expression for the magnetic field due to the straight wire at a point that does not lie on the perpendicular bisector of the wire.

Source Coordinates:

In order to apply the Biot-Savart, we choose a Cartesian coordinates system with the $x$-axis aligned along the wire, with the origin at the center of the wire.

**Question 1:** With this choice of coordinate system, choose an infinitesimal current element located anywhere along the $x$-axis except at the ends of the wire. What is the position vector for this infinitesimal current element, $\mathbf{r}'$? What is your integration variable? What are the limits of your integration variable? Write down a vector expression for the infinitesimal current element $I d\mathbf{s}'$. 
Field Point Coordinates

The next step is to identify the location of your field point in terms of your coordinate system.

**Question 2:** Write down a vector expression for the position vector \( \mathbf{r} \) from the origin to the field point.

**Question 3:** What is the distance \( r \) from the source to the field point?

**Question 4:** The unit vector, \( \hat{\mathbf{r}} \), located at the field point, has magnitude, \( |\hat{\mathbf{r}}|=1 \), and points from the source to the field point. Write down a vector expression for the unit vector \( \hat{\mathbf{r}} \).
Biot-Savart Law:

We can now apply the Biot-Savart Law to calculate the contribution of the infinitesimal current source, \( I d\mathbf{s} \), to the infinitesimal magnetic field, \( d\mathbf{B} \), at the field point \( P \).

**Question 5:** Write down a vector expression in terms of your choice of source and field point coordinates for the contribution of the infinitesimal current source, \( I d\mathbf{s} \), to the infinitesimal magnetic field,

\[
d\mathbf{B} = \frac{\mu_0 I d\mathbf{s} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.
\]

**Question 6:** If you have not already done so, explicitly calculate the cross product that appears in the above equation and find a vector expression for the infinitesimal magnetic field,

The magnetic field at the point \( P \) is given by the integral over the wire

\[
\mathbf{B} = \int_{\text{wire}} d\mathbf{B}.
\]

**Question 7:** What are the endpoints of this integral?
Question 8: Set up the integral expression for the magnetic field at the field point based on your results up this point. (Can you integrate this?)

Question 9: Perform the integration, be careful with the limits and find an expression for the magnetic field as a function of the coordinates \((x,y)\) of the field point \(P\).

Question 10: What is your answer for the important limiting case when the length of the wire approaches infinity, \(L \to \infty\)?
Problem 2: Archimedes Spiral

A wire segment is bent into the shape of an Archimedes spiral (see sketch). The equation that describes the curve in the range $0 \leq \theta \leq \pi$ is

$$r(\theta) = a + \frac{b}{\pi} \theta,$$

where $\theta$ is the angle from the $x$-axis in radians. The point $P$ is located at the origin of our $xy$ coordinate system. The vectors $\hat{e}_r$ and $\hat{e}_\theta$ are the unit vectors in the radial and azimuthal directions, respectively, as shown. The wire segment carries current $I$, flowing in the sense indicated.

![Diagram of Archimedes Spiral and coordinate system]

What is the magnetic field at point $P$?