Problem Solving 10: Plane Electromagnetic Waves, Poynting Energy Flow, and Interference

OBJECTIVES

1. To introduce the concept of energy flow through space due to plane sinusoidal traveling electromagnetic waves.

2. To quantify that energy flow by introducing the Poynting vector.

3. To quantify the concept of radiation pressure.

4. To develop an order of magnitude feel for the energy flow from the sun into the earth’s atmosphere

5. To understand the meaning of constructive and destructive interference

6. To understand how to determine the interference conditions for double slit interference

7. To understand how to apply the interference conditions for diffraction

REFERENCE: Course Notes: Sections 13-1-13-7, 14-1 through 14-7, 14.9
Problem 1: *Plane Electromagnetic Waves* Consider a sinusoidal plane electromagnetic wave given by the expression

\[ \mathbf{E}(z,t) = E_x(z,t) \hat{i} = E_0 \cos(kz + \omega t) \hat{i}. \]

**Question 1:** What is the direction of propagation of this electromagnetic plane wave?

**Answer:**

**Question 2:** Determine the relation between \( \omega \) and \( k \) for this field to satisfy Maxwell’s wave equation for the electric field, \( \frac{1}{c^2} \frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial z^2} \)? Hint: calculate both \( \frac{\partial^2 E_x}{\partial t^2} \) and \( \frac{\partial^2 E_x}{\partial z^2} \).

**Answer:**
**Question 3:** Find an expression for the direction and magnitude of the magnetic field associated with this plane electromagnetic wave.

**Answer:**

**Question 4:** What is the Poynting vector $\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ associated with this wave?

**Answer:**

**Question 5:** The definition of a time average of a periodic function $f(t)$ over one period is given by the integral expression

$$\langle f(t) \rangle = \frac{1}{T} \int_{0}^{T} f(t) \, dt .$$

The time-averaged value of the square of the sine function is

$$\left\langle \sin^2 \left( \omega t + \frac{\Phi}{2} \right) \right\rangle = \frac{1}{2} .$$

What is the time-average of the Poynting vector $\langle \mathbf{S} \rangle$? Recall that $c^2 = \frac{1}{\mu_0 \varepsilon_0}$.

**Answer:**
Problem 2 Poynting Vector and Radiation Pressure

Introduction

A plane electromagnetic wave transports energy in the direction of propagation of the wave. The power per square area is given by the Poynting vector

\[ \overrightarrow{S} = \frac{\overrightarrow{E} \times \overrightarrow{B}}{\mu_0}. \]

The power that flows into the rectangular volume cross-sectional area \( A \) and length \( c\Delta t \) appears as rate of change of the energy stored in the fields inside the volume.

\[ P_{\text{power}} = \left( \overrightarrow{S} \right) \cdot A = \frac{d}{dt} \langle U_{\text{total}} \rangle. \]

The electromagnetic wave also transports momentum, and hence can exert a radiation pressure on a surface due to the absorption and reflection of the momentum.

The momentum carried by an electromagnetic wave is related to the energy of the wave according to

\[ U = c|\overrightarrow{p}|. \]

If the plane electromagnetic wave is completely absorbed by a surface of cross-sectional area \( A \) then the momentum \( |\Delta \overrightarrow{p}| \) delivered to the surface in a time \( \Delta t \) is given by

\[ |\Delta \overrightarrow{p}| = \frac{\Delta U}{c}. \]

The force that the wave exerts on the surface is then the rate of change of the momentum in time

\[ F = \lim_{\Delta t \to 0} \frac{|\Delta \overrightarrow{p}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{c} \frac{\Delta U}{\Delta t} = \frac{1}{c} \frac{dU}{dt}. \]

Since the rate of change of energy is related to the power flowing across the surface, the force is

\[ F = \frac{1}{c} \frac{dU}{dt} = \frac{1}{c} P_{\text{power}} = \frac{1}{c} \left\langle \overrightarrow{S} \right\rangle \cdot A. \]

The radiation pressure \( P_{\text{pressure}} \) is then defined to be the force per area that the wave exerts on the surface.
\[
\begin{align*}
\text{P}_{\text{pressure}}^{\text{abs}} & \equiv \frac{F}{A} = \frac{1}{c} \langle |\mathbf{s}| \rangle, \quad \text{perfectly absorbing.}
\end{align*}
\]

When the surface completely reflects the wave, then the change in momentum is twice the absorbing case since the wave completely reverses direction,

\[
|\Delta \mathbf{p}| = 2 \frac{\Delta U}{c}.
\]

Therefore the radiation pressure of a wave on a perfectly reflecting surface is

\[
\text{P}_{\text{pressure}}^{\text{ref}} \equiv \frac{F}{A} = 2 \frac{1}{c} \langle |\mathbf{s}| \rangle, \quad \text{perfectly reflecting.}
\]

**Radiation Pressure: Solar Sail**

Suppose you want to ‘sail’ a space capsule through the solar system by using the force due to the ‘radiation pressure’ from the sun’s light to balance the gravitational attraction from the sun. Imagine that the mass of the capsule and the sail is 1000 kg, that the circular sail is perfectly reflective, and that it is oriented face-on to the sun. The luminosity of the sun is \( P_{\text{sun}} = 4 \times 10^{26} \text{ J/sec} \) or alternatively the solar constant is \( 1.4 \times 10^3 \text{ W} \cdot \text{m}^{-2} \) at the top of the earth’s atmosphere. The mass of the sun is \( 1.99 \times 10^{30} \text{ kg} \). Note all of the above quantities are time-averaged.

**Question 1:** At a distance \( r \) from the sun, the total power output of the sun is spread over a spherical area \( 4\pi r^2 \). Determine the time-averaged value of the magnitude of the Poynting vector on the spherical surface at a distance \( r \) from the sun.

**Answer:**
**Question 2:** Determine the radiation pressure on the solar sail at a distance $r$ from the sun. You may assume that the solar sail is a perfectly reflecting object.

**Answer:**

**Question 3:** Determine the radiation force on the solar sail $A$ that has area at a distance $r$ from the sun.

**Answer:**

**Question 4:** The minimum sail area necessary to balance the radiation force and the attractive gravitational force can be found by setting $F_{\text{rad}}(r) = F_{\text{grav}}(r)$. Using the universal law of gravity (magnitude given by $F_{\text{grav}}(r) = \frac{G m_{\text{sun}} m}{r^2}$). What is the minimum area for the sail in order to exactly balance the gravitational attraction from the sun?

**Answer:**
**Question 5:** Does the minimum area for the sail depend on how far the sail is away from the sun? Explain your reasoning.

**Answer:**
Problem 3 Interference

Introduction

The *Huygens Principle* states that every unobstructed point on a wavefront will act a source of a secondary spherical wave. We add to this principle, the *Superposition Principle* that the amplitude of the wave at any point beyond the initial wave front is the superposition of the amplitudes of all the secondary waves.

![Figure 1: Huygens-Fresnel Principle applied to double slit](image)

When ordinary light is emitted from two different sources and passes through two narrow slits, the plane waves do not maintain a constant phase relation and so the light will show no interference patterns in the region beyond the openings. In order for an interference pattern to develop, the incoming light must satisfy two conditions:

- The light sources must be coherent. This means that the plane waves from the sources must maintain a constant phase relation.
- The light must be monochromatic. This means that the light has just one wavelength.

When coherent monochromatic laser light falls on two slits separated by a distance $d$, the emerging light will produce an interference pattern on a viewing screen a distance $D$ from the center of the slits. The geometry of the double slit interference is shown in the figure below.
Consider light that falls on the screen at a point $P$ a distance $y$ from the point $O$ that lies on the screen a perpendicular distance $D$ from the double slit system. The light from the slit 2 will travel an extra distance $r_2 - r_1 = \Delta r$ to the point $P$ than the light from slit 1. This extra distance is called the path length.

**Question 1:** Draw a picture of two traveling waves that add up to form *constructive interference*.

**Answer:**

**Question 2:** Draw a picture of two traveling waves that add up to form *destructive interference*.

**Answer:**
**Question 3:** Explain why constructive interference will appear at the point $P$ when the path length is equal to an integral number of wavelengths of the monochromatic light.

\[ \Delta r = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ constructive interference} \]

**Answer:**

**Question 4:** Based on the geometry of the double slits, show that the condition for constructive interference becomes

\[ d \sin \theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ constructive interference.} \]

**Answer:**

**Question 5:** Explain why destructive interference will appear at the point $P$ when the path length is equal to an odd integral number of half wavelengths

\[ d \sin \theta = \left( m + \frac{1}{2} \right)\lambda, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots \text{ destructive interference.} \]

**Answer:**
**Question 6:** Let $y$ be the distance between the point $P$ and the point $O$ on the screen. Find a relation between the distance $y$, the wavelength $\lambda$, the distance between the slits $d$, and the distance to the screen $D$ such that a constructive interference pattern will occur at the point $P$.

**Answer:**

**Question 7:** Find a similar relation such that destructive interference fringes will occur at the point $P$.

**Answer:**