A Periodic Delta-"Function"

as a

Sum of Complex Exponentials

An object under consideration is

$$\lim_{N\to\infty} \sum_{k=-N}^{N} e^{2\pi i\, kt.}$$

This is refered to as an "object" because the limit does not exist as a function, but is rather a "distribution." That is, a proposed equivalent form is

$$\lim_{N\to\infty} \sum_{k=-N}^{N} e^{2\pi i\, kt} = \sum_{n=-\infty}^{\infty} \delta\left(n-t\right),$$

and like all δ-"functions," this object must be judged by what it does to other functions, as opposed to any explicit functional form.

For the purposes of examining the behavior as $N \to \infty$, consider the partial sum

$$F_N(t) = \sum_{k=-N}^{N} e^{2\pi i\, kt}.$$

Several properties of $F_N(t)$ are immediately clear:

- $F_N(t)$ is perodic with period 1,
- $F_N(t)$ is an even function of t, and
- $F_N(0) = 2N + 1$.

As a consequence of the first property, the remainder of these notes will consider only the range $-\frac{1}{2} \leq t \leq \frac{1}{2}$.

It should be clear that $F_N(t)$ is a geometric series, and so may be found in closed form;

$$
\begin{aligned}
F_N(t) &= \sum_{k=-N}^{N} e^{2\pi i\, kt} \\
&= e^{-2\pi i\, Nt} \sum_{m=0}^{2N} \left(e^{2\pi i\, t}\right)^m \\
&= e^{-2\pi i\, Nt}\, \frac{e^{2\pi i(2N+1)\, t} - 1}{e^{2\pi i\, t} - 1} \\
&= e^{-2\pi i\, Nt}\, \frac{e^{\pi i(2N+1)\, t}\left[e^{\pi i(2N+1)\, t} - e^{-\pi i(2N+1)\, t}\right]}{e^{\pi i\, t}\left[e^{\pi i\, t} - e^{-\pi i\, t}\right]} \\
&= \frac{\sin\left((2N+1)\pi\, t\right)}{\sin\left(\pi\, t\right)}
\end{aligned}
$$

for t non-integral. For t integral, the periodicity of $F_N(t)$ gives $F_N(t) = 2N + 1$. The nature of $F_N(t)$ may be examined by simple plots; a MAPLE worksheet may be downloaded from the **8.03-ESG** web page.

To see how $F_N(t)$ might be considered a δ-distribution in the limit $N \to \infty$, consider

$$
\int_{-1/2}^{1/2} F_N(t)\, dt = 2 \int_0^{1/2} \frac{\sin\left((2N+1)\pi\, t\right)}{\sin\left(\pi\, t\right)}\, dt.
$$

Make the substitution

$$
u = (2N+1)\pi\, t, \qquad dt = \frac{du}{(2N+1)\pi}\,,
$$

so that

$$
\int_{-1/2}^{1/2} F_N(t)\, dt = 2 \int_0^{(N+1/2)\pi} \frac{\sin u}{(2N+1)\pi\, \sin\left(u/(2N+1)\right)}\, du.
$$

In the limit as $N \to \infty$, the upper limit of the integral goes to infinity, giving an improper definite integral, while the denominator of the integrand approaches $\pi\, u$, as may be checked using l'Hôpital's Rule. The result is that

$$
\int_{-1/2}^{1/2} F_N(t)\, dt \qquad \longrightarrow \qquad \frac{2}{\pi} \int_0^{\infty} \frac{\sin u}{u}\, du.
$$

This integral is well-known to many, but may be derived by use of a *Tauberian Parameter*. What follows is from the 18.023 texts, *Calculus: an Introduction to Applied Mathematics* by Greenspan, Benney and Turner, Page 533, or *Calculus: an Introduction to Applied Mathematics* by Greenspan and Benney, Page 514. Specifically, consider the improper definite integral

$$
\int_0^{\infty} e^{-a\, u}\, \sin u\, du, \qquad a > 0.
$$

This integral may be done by many means, both foul and fair, to obtain $\dfrac{1}{a^2+1}$. Then,

$$\int_0^\infty \frac{e^{-a\,u}\,\sin u}{u}\,du = -\int da\left[\int_0^\infty e^{-a\,u}\,\sin u\,du\right] + C$$
$$= C - \tan^{-1}(a).$$

The constant of integration is found by taking $a \to \infty$, in which case the integral with respect to u on the left above must be 0, and so $C = \frac{\pi}{2}$, and taking $a \to 0$,

$$\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2}.$$

Thus, we have

$$\lim_{N\to\infty} \int_{-1/2}^{1/2} F_N(t)\,dt = 1,$$

a necessary candidate for any representation of the δ-distribution.

Now, the fact is that the above representation does not have the property that the limit converges to 0 uniformly as $N \to 0$ for nonintegral t. That is, consider the first maximum of $|F_N(t)|$ (or, equivalently, $(F_n(t))^2$) that occurs at a value of $t > 0$. It can be shown by basic calculus that this value of t, call it t_{max}, is the least positive solution of the transcendental (or is it?) equation

$$\tan(\pi\,t) = \frac{1}{2N+1}\,\tan\left((2N+1)\pi\,t\right).$$

This solution may be found numerically for a given N, or graphically. The advantage to the graphical solution is that it is seen that as $N \to \infty$, the least positive value of t that is an intersection of the graphs of the two functions is at $(2N+1)\pi\,t \sim \frac{3\pi}{2}$, giving $t_{\mathrm{max}} \sim \frac{3}{4N}$.

Two such graphs, for $N = 5$, are given in plots linked from the page that gave these notes. In the first, with the expanded scale, the intersection at $t \sim 0.13$ can be seen (MAPLE gives this point as 0.1304, to larger precision if desired), similar to the crudely predicted value of $3/22 \sim 0.1364$. The second plot shows that the successive extrema occur essentially at the places where the graph of $\dfrac{\tan(11\,\pi\,t)}{11}$ diverges. (MAPLE's plotting algorithm results in having these asymptotes more or less drawn in by default.)

For N large, then,

$$F_N\left(t_{\mathrm{max}}\right) \sim -\frac{4N}{3\pi}, \qquad \left|\frac{F_N(0)}{F_N\left(t_{\mathrm{max}}\right)}\right| \sim \frac{3\pi}{2}.$$

We can interpret this as having the "peaks" in $F_N(t)$ be roughly proportional to N, but with alternating signs, and coming closer together as N becomes large. (This property is also seen from the cited MAPLE worksheet.) Thus, any integral of $F_N(t)$, mulitplied by any well-behaved fuction, will have any contributions away from the principle peaks at integral t "wash out," leaving only the area in the region of the principle peaks, shown above to be 1, times the value of the function at these values. This is indeed the desired property of the δ-"function."

To be slightly more rigorous, consider the prior derivation of the integral over one period, but multiplied by a well-behaved (for our purposes, continuous) function $f(t)$;

$$\begin{aligned}
\int_{-1/2}^{1/2} f(t)\, F_N(t)\, dt &= 2\int_0^{1/2} f(t)\, \frac{\sin\left((2N+1)\pi\, t\right)}{\sin\left(\pi\, t\right)}\, dt \\
&= 2\int_0^{(N+1/2)\pi} f\left(u/(2N+1)\pi\right)\, \frac{\sin u}{(2N+1)\pi\, \sin\left(u/(2N+1)\right)}\, du \\
&\to \frac{2}{\pi}\int_0^{\infty} f(0)\, \frac{\sin u}{u}\, du \\
&= f(0)\, \frac{2}{\pi}\int_0^{\infty} \frac{\sin u}{u}\, du \\
&= f(0),
\end{aligned}$$

where the same substitution, $u = (2N+1)\pi\, t$, has been made. This shows that $\lim_{N\to\infty} F_N(t)$ has the proper integral property, even though F_N does not converge uniformly to 0 for nonintegral t.