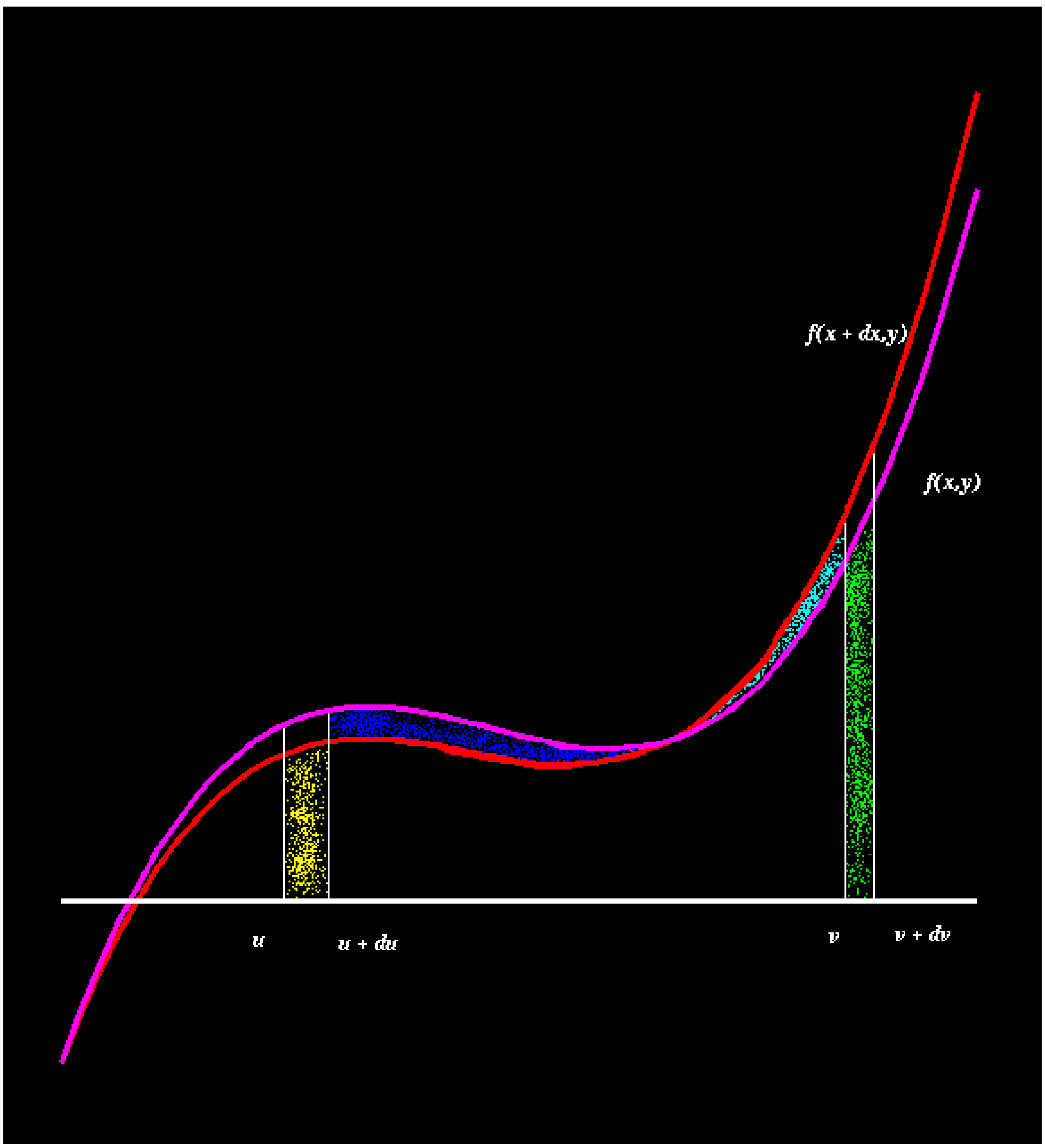
Leibniz's Formula

For these notes, the notation will be that of *Simmons*, and all page and equation references are to that volume.

When the Web gets better, all typefaces will be the same. Until then, the font in the figure uses a pointy-bottom "vee" that looks far too much like the Greek letter "nu" (ν). We can deal with this. When it becomes easier to put math on the Web, this text could be inline with the figure, but that may not be a great advantage. Anyway ...



As stated in Problem 19 of Section 19.6, *Leibniz's Formula* is

$$\frac{d}{dx}\int_u^v f(x,\, y)\, dy = -f(x,u)\frac{du}{dx} + f(x,v)\frac{dv}{dx} + \int_u^v \left[\frac{\partial}{\partial x} f(x,\, y)\right]\, dy.$$

As indicated in the text, the first two terms follow directly from the Fundamental Theorem of Calculus, and the third is derived in *Simmons*, Appendix A.18, Pages 838-839. These notes will not reproduce this proof, but merely offer a graphical interpretation.

In the figure, the horizontal axis is the y-axis (axis labels are not shown to avoid too much clutter), and the magenta (or violet) curve represents $f(x,\, y)$. What is happening is that each value of x gives a different function of y. We could regard $f(x,\, y)$ as a surface which we are viewing from along the x-axis, and each value of x gives a "slice" of this surface, and hence a curve. The dark red curve is $f(x+dx,\, y)$, that is, the curve as a function of y for a slightly different value of x.

The value of

$$\int_u^v f(x,\, y)\, dy$$

is then the area above the horizontal line and beneath the magenta curve, between the lower limit of u and the upper limit of v.

In the full form of Leibniz's Rule as given above, the limits u and v are functions of x, and will in general change as x changes. So, when x changes, there are *three* contributions to the change of the integral:

(1) The lower limit changes. In the figure, du/dx is positive, and the area under the red curve has been *decreased* by an amount equal to the area of the yellow-speckled region, the product of du and $f(x,\, u)$. Denote this area as

$$dA_1 = -du\, f(x,\, u) = -dx\, \frac{du}{dx}\, f(x,\, u).$$

Before continuing, it may be a good idea to explain why the height of the rectangle is taken to be $f(x,\, u)$ instead of $f(x,\, u + du)$, $f(x + dx,\, u)$ or even $f(x + dx,\, u + du)$. Using any of these values would put the upper edge of the yellow rectangle at another of the vertices of the small unshaded region above the yellow region and between the red and magenta curves. This possible variation in the area is second-order in dx, and will not be considered. A useful analogy is the situation

$$\begin{aligned} d\left(z^2\right) &= (z + dz)^2 - z^2 \\ &= 2z\, dz + dz^2 \\ &= 2z\, dz; \end{aligned}$$

in increasing the length of the side of a square by a small amount, the differential of the area neglects the small square of size dz^2.

(2) The upper limit changes. By a similar reason (again, dv is positive in the figure),

$$dA_2 = dv\, f(x,\, v) = dx\, \frac{dv}{dx}\, f(x,\, v).$$

(3) The integrand changes. In the figure, the change in the area due to the change in x is indicated by the blue-speckled area between the curves. The area between the curves is

$$\int_u^v \left[f(x + dx,\, y) - f(x,\, y)\right]\, dy.$$

In the figure, note carefully that $f(x + dx,\, y) - f(x,\, y)$ is negative for part of the region $u \le y \le v$ and positive for the other part (and of course 0 at a point). The contribution to the change in the area for $f(x + dx,\, y) < f(x,\, y)$ is in dark blue, the contribution for $f(x + dx,\, y) > f(x,\, y)$ in light blue. The total contribution is

$$dA_3 = \int_u^v \left[f(x + dx,\, y) - f(x,\, y)\right]\, dy = \int_u^v \frac{\partial f}{\partial x}\, dx\, dy = dx \int_u^v \frac{\partial f}{\partial x}\, dy,$$

where the differential dx, which cannot depend on the integration variable dy, has been taken out of the integral. (In last two integrals in the above expression, the dependence of $\frac{\partial f}{\partial x}$ on x and y has been supressed.)

The net change in area is then

$$\begin{aligned} dA &= dA_1 + dA_2 + dA_3 \\ &= -dx\, \frac{du}{dx}\, f(x,\, u) + dx\, \frac{dv}{dx}\, f(x,\, v) + dx \int_u^v \frac{\partial f}{\partial x}\, dy \\ &= dx \left[-\frac{du}{dx}\, f(x,\, u) + \frac{dv}{dx}\, f(x,\, v) + \int_u^v \frac{\partial f}{\partial x}\, dy\right], \end{aligned}$$

from which Leibniz's Formula follows.