Quantum Physics III (8.06) Spring 2003
FINAL EXAMINATION
Tuesday May 20, 1:30 pm
You have 3 hours.

There are 8 problems, totalling 180 points. Do all problems.

Answer all problems in the blue books provided.
Write YOUR NAME on EACH blue book you use.
Budget your time wisely, using the point values as a guide. Note also that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

- Conservation of Probability

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \vec{\nabla}\cdot\vec{J}(\vec{r},t) = 0$$

where

$$\rho(\vec{r},t) = |\psi(\vec{r},t)|^2 \ ; \qquad \vec{J}(\vec{r},t) = \frac{\hbar}{2im}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

- Operators for Spin-1/2 particle

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ; \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$[\hat{x},\hat{p}] = i\hbar \ .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) \ ,$$

and satisfy

$$[\hat{a},\hat{a}^\dagger] = 1 \ .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \ .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

- Spherical Coordinates

$$x = r\sin\theta\cos\phi \ ; \quad y = r\sin\theta\sin\phi \ ; \quad z = r\cos\theta$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

- Angular Momentum

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \ ; \quad [\hat{L}^2, \hat{L}_i] = 0$$

$$\hat{L}^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle \ ; \quad \hat{L}_z|\ell, m\rangle = \hbar m|\ell, m\rangle$$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$[\hat{L}_\pm, \hat{L}^2] = 0 \ ; \quad [\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$$

$$\hat{L}^2 = \hat{L}_+\hat{L}_- + L_z^2 - \hbar L_z$$

$$\hat{L}_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle$$

- Angular momentum operators in spherical coordinates

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \ ; \quad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle\theta,\phi|\ell,m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \ ; \quad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi) \ ; \quad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi) \ ; \quad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi) \ ;$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

- Some useful constants:

$$\hbar c = 197 \times 10^{-7} \text{ eV cm}$$

The mass of the electron is $m_e = 0.511$ MeV/c^2. If B is 1 gauss, then the force eB is 300 eV/cm.

- Particle in an Electric and/or Magnetic Field:

  The Hamiltonian for a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and an electric field $\vec{E} = -\vec{\nabla}\phi$ is:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \qquad (1)$$

- Gauge invariance:

  If $\psi(\vec{x},t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x},t) = \exp\left(\frac{ie}{\hbar c} f(\vec{x},t)\right)\psi(\vec{x},t)$$

  solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla} f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

  Suppose that

$$H = H^0 + H'$$

  where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0:

$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \ .$$

  Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^0 + H'_{nn} + \sum_{m\neq n}\frac{|H'_{nm}|^2}{E_n^0 - E_m^0} + \ldots$$

$$|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m\neq n}\frac{H'_{mn}}{E_n^0 - E_m^0}|\psi_m^0\rangle + \ldots$$

  where $H'_{nm} \equiv \langle\psi_n^0|H'|\psi_m^0\rangle$.

- Connection Formulae for WKB Wave Functions:

  At a turning point at $x = a$ at which the classically forbidden region is at $x > a$:

$$\frac{2}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_x^a p(x')dx' - \frac{\pi}{4}\right] \quad \leftarrow \quad \frac{1}{\sqrt{\kappa(x)}}\exp\left[-\frac{1}{\hbar}\int_a^x \kappa(x')dx'\right]$$

$$\frac{1}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_x^a p(x')dx' + \frac{\pi}{4}\right] \quad \rightarrow \quad \frac{1}{\sqrt{\kappa(x)}}\exp\left[+\frac{1}{\hbar}\int_a^x \kappa(x')dx'\right]$$

  At a turning point at $x = b$ at which the classically forbidden region is at $x < b$:

$$\frac{1}{\sqrt{\kappa(x)}}\exp\left[-\frac{1}{\hbar}\int_x^b \kappa(x')dx'\right] \quad \rightarrow \quad \frac{2}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_b^x p(x')dx' - \frac{\pi}{4}\right]$$

$$\frac{1}{\sqrt{\kappa(x)}}\exp\left[+\frac{1}{\hbar}\int_x^b \kappa(x')dx'\right] \quad \leftarrow \quad \frac{1}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_b^x p(x')dx' + \frac{\pi}{4}\right]$$

- Born Approximation to Scattering Amplitude:

$$f(\theta,\phi) = f(\vec{q}) = -\frac{m}{2\pi\hbar^2}\int d^3r\, \exp(-i\vec{q}\cdot\vec{r})V(\vec{r})$$

where $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transfer.

- Partial Wave Analysis of Scattering from a spherically symmetric potential:

$$f(\theta) = \sum_l (2\ell+1)P_l(\cos\theta)\frac{\exp(2i\delta_\ell)-1}{2ik}$$

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

- Time Dependent Perturbation Theory:

  Consider a system with the Hamiltonian

$$H = H_0 + V(\vec{r})\cos(\omega t)\exp(-t^2/T^2)$$

and denote the matrix element of V between eigenstates of H_0 named $|a\rangle$ and $|b\rangle$ by V_{ab}. Then, in the large T limit, if the system is initially in state $|a\rangle$, the probability that it is in the state $|b\rangle$ for $t \to +\infty$ is:

$$P_{a\to b} = \frac{\pi}{2\hbar^2}|V_{ab}|^2\delta(\omega-\omega_{ab})\sqrt{\pi}T$$

where $\hbar\omega_{ab} = E_b - E_a$.

1. **Short Answer Questions (40 points)**

   (a) (5 points) Consider an infinite solenoid, containing magnetic flux Φ. For what values of Φ (if any) are the energy eigenvalues of a quantum mechanical particle constrained to stay far from the solenoid exactly the same as they would be if Φ were zero?

   (b) (5 points) What are the units of pressure in natural units? (Your answer should be of the form $\mathrm{eV}^p\ \hbar^q\ \mathrm{c}^r$.)

   (c) (5 points) Consider a hydrogen molecule, made of two protons and two electrons. Denote the separation between the two protons by R. Describe how you would construct the effective potential $V(R)$ that describes the vibrations of the molecule.

   (d) (5 points) The quantum of conductance observed in the Integer Quantum Hall Effect is a combination of fundamental constants. What is it? [No derivation required. And, you need not get the numerical coefficient right to get full credit.]

   (e) (10 points) Consider a spin one-half particle with magnetic moment in a slowly varying magnetic field $\vec{B}(t)$. The Hamiltonian of the system is

   $$H = -\frac{2\mu_0}{\hbar}\vec{S}\cdot\vec{B}(t)\ .$$

   Assume that $\vec{B}(t)$ varies slowly enough that the adiabatic theorem applies. Assume that the magnitude of $\vec{B}$ is time-independent; only the direction of $\vec{B}$ changes with time. Give a precise definition of Berry's phase for this problem, and a precise statement of what determines its value.

   (f) (10 points) Consider a particle with mass m in a one-dimensional simple harmonic oscillator potential $V = (m\omega_0^2/2)x^2$. At early times ($t \to -\infty$) the particle is in the harmonic oscillator ground state. This particle is subject to a time dependent perturbation

   $$H' = \epsilon x^4 \cos(\omega t)\exp(-t^2/T^2)$$

   where T is much greater than $1/\omega$ and $1/\omega_0$, and where ϵ is very small. For what value(s) of the perturbing frequency ω will there be a nonzero probability that the system makes a transition out of the ground state of the harmonic oscillator potential? For each such frequency, to which excited state(s) of the harmonic oscillator are transitions possible?

2. **The Landau Problem (15 points)**

   Consider an electron in a uniform magnetic field $\vec{B} = (0, 0, B_0)$. Assume the electron is restricted to move in the xy-plane.

   (a) (5 points) Write the Hamiltonian in a gauge in which p_y commutes with H.

   (b) (5 points) Derive the energy eigenvalues.

   (c) (5 points) Find the relationship between $\langle x \rangle$ and $\langle p_y \rangle$ in an energy eigenstate.

3. **A Resonant Adiabatic Transition (15 points)**

   Consider a two-state system, whose basis states we denote $|1\rangle$ and $|2\rangle$. The Hamiltonian for the system is time dependent. It is given (in the $\{|1\rangle, |2\rangle\}$ basis) by

$$H(t) = \begin{pmatrix} \alpha t & \Delta \\ \Delta & -\alpha t \end{pmatrix} \tag{2}$$

   where α and Δ are constants.

   (a) (5 points) What are the eigenvalues of $H(t)$? Sketch the two eigenvalues as a function of t.

   (b) (5 points) What is the eigenstate associated with the lower eigenvalue for $t \to -\infty$? What is the eigenstate associated with the lower eigenvalue at $t = 0$? What is the eigenstate associated with the lower eigenvalue for $t \to +\infty$?

   (c) (5 points) What is the condition on α such that the adiabatic theorem applies to this problem? If this condition is satisfied, and if the system starts in the state $|1\rangle$ at $t \to -\infty$, what will the state of the system be at $t \to +\infty$?

4. **The Bouncing Ball (15 points)**

   Consider a quantum mechanical particle moving in one dimension, whose motion is restricted to $x \geq 0$ by the requirement that ψ vanishes at $x = 0$. The particle feels a potential $V(x) = mgx$ (for $x > 0$).

   Use the WKB approximation throughout this problem.

   (a) (2 points) Consider a wave function with energy $E > 0$. What is the allowed region of x for a solution with energy E? What is the forbidden region?

   (b) (7 points) Using WKB connection formulae given on the information sheets, derive the quantization condition appropriate for this potential.

   (c) (6 points) Find the energy eigenvalues.

5. **Initializing the Grover Algorithm (10 points)**

   Consider the 4 dimensional Hilbert space formed by taking the tensor product of the Hilbert spaces for two spin-one-half particles.

   We denote the basis states as follows:

$$
\begin{aligned}
|0\rangle &= |0,0\rangle \\
|1\rangle &= |0,1\rangle \\
|2\rangle &= |1,0\rangle \\
|3\rangle &= |1,1\rangle
\end{aligned}
$$

   where, for example, $|1,0\rangle$ means a state in which all two spins are in eigenstates of S_z, with eigenvalues $-\hbar/2$, $+\hbar/2$.

   You should work in a basis with basis vectors ordered as above.

   Construct a unitary operator U such that

$$U|3\rangle = |s\rangle$$

   where the state $|s\rangle$ is given by

$$|s\rangle = \frac{1}{\sqrt{4}}\Big[|0\rangle + |1\rangle + |2\rangle + |3\rangle\Big] \ .$$

   Write the 4×4 matrix U as the product of two 4×4 unitary matrices each of which acts only within the Hilbert space of one of the two spins.

6. **Two Unrelated Variational Proofs (25 points)**

   (a) (5 points) Prove that, in time-independent perturbation theory, the sum of the second order, third order, fourth order plus all higher order corrections to the ground state energy of a quantum system must be either zero or negative.

   (b) (20 points) Consider a particle of mass m in a one-dimensional potential. The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x) \ .$$

   The potential has the following properties: $V = 0$ for $x \geq b$ and for $x \leq -b$; $V < 0$ for $-b < x < b$. Use the variational principle to prove that there is at least one bound state solution.

   [Hint: try $\psi(x) \propto \exp(-x^2/a^2)$. It is in fact not necessary to evaluate any dimensionless integrals in order to construct a proof. In case you want them, though, here are some integrals: $\int_{-\infty}^{\infty} dx \exp(-x^2) = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} dx\, x^2 \exp(-x^2) = \sqrt{\pi}/2$.]

7. **Low Energy Scattering off a Spherical Hole (30 points)**

A particle of mass m and momentum $\hbar k$ scatters from a potential hole in 3-dimension. $V = -V_0$ for $r \leq R$ and $V = 0$ for $r \geq R$.

In the limit of small V_0 and low energy, the differential cross-section can be calculated in two different ways.

(a) (10 points) Recall that the radial wave function $u_0(r) = R_0(r)/r$ satisfies the one-dimensional Schrödinger equation

$$-u_0'' + \frac{2mV(r)}{\hbar^2} u_0 = k^2 u_0$$

with $u_0(0) = 0$. If V_0 were zero, $u_0 \propto \sin kr$. With V_0 nonzero, there is a phase shift so that for $r > R$ we have $u_0 \propto \sin(kr + \delta_0)$.

Assume that kR is so small that only the s-wave contributes to the scattering. Furthermore, assume that the s-wave phase shift $\delta_0(k)$ is given by

$$\delta_0(k) = -ka \ .$$

Show that the scattering length a is given by

$$a = R - \frac{\tan(\gamma R)}{\gamma} \tag{3}$$

where $\gamma^2 = \frac{2mV_0}{\hbar^2}$.

[Note: because part (a) is posed as "Show that ..." you can do all subsequent parts even if you do not succeed in doing part (a).]

(b) (4 points) What is the differential cross section, $d\sigma/d\Omega$, in terms of the scattering length a?

(c) (3 points) What is special about the values of γ at which $a \to \infty$?

(d) (3 points) What is the Ramsauer-Townsend effect and what condition must γ satisfy for it to occur?

(e) (6 points) If V_0 is small, the Born approximation can be applied. Evaluate the differential cross-section using the Born approximation and assuming that kR is small.

(f) (4 points) Plug the expression (3) for a into your result for the differential cross section obtained in part (b), and simplify the resulting expression upon assuming that V_0 is small. Show that you obtain the same result as in part (e). [Hint: $\tan x = x + \frac{1}{3}x^3 + \ldots$.]

8. **Particle on a Sphere, with Three Different Perturbations (30 points)**

Note: spherical harmonics that you will need for this problem are given on the information sheets. Integrals that you might need are given below.

Consider a particle with mass M restricted to move on a sphere with radius $r = 1$. The wave function for the particle is thus a function only of the angles θ and ϕ.

In the absence of any potential, the Hamiltonian for this system is

$$H^0 = \frac{p^2}{2M} = \frac{L^2}{2Mr^2} = \frac{L^2}{2M} \, .$$

The particle has charge q. The particle has no spin.

(a) (3 points) What are the energies and degeneracies of the states with angular momentum quantum number $\ell = 0, 1, 2$?

(b) (5 points) Now, turn on a magnetic field in the z-direction $\vec{B} = B_0 \hat{z}$. To first order in B_0, the term added to the Hamiltonian as a result of the magnetic field is given by

$$H' = -\frac{q}{2Mc} \vec{B} \cdot \vec{L}$$

Calculate the energies of the $\ell = 2$ state(s) to first order in B_0.

(c) (10 points) Turn the magnetic field from part (b) off. Now, turn on a perturbation described by the Hamiltonian

$$H' = \epsilon(x^2 - y^2)$$

Calculate the energies of the $\ell = 1$ state(s) to first order in ϵ.

(d) (12 points) Turn off the perturbation from part (c). Now, turn on an electric field in the x-direction, $\vec{E} = E_0 \hat{x}$. Calculate the energies of the $\ell = 0$ state(s) to *second* order in E_0.

Here are some integrals, some of which you might find useful:

$$\begin{aligned}
\int_0^\pi d\theta \sin\theta &= 2 \\
\int_0^\pi d\theta \sin^2\theta &= \frac{\pi}{2} \\
\int_0^\pi d\theta \sin^3\theta &= \frac{4}{3} \\
\int_0^\pi d\theta \sin^4\theta &= \frac{3\pi}{8} \\
\int_0^\pi d\theta \sin^5\theta &= \frac{16}{15} \\
\int_0^\pi d\theta \sin^6\theta &= \frac{5\pi}{16}
\end{aligned}$$

$$
\begin{aligned}
\int_0^\pi d\theta \sin\theta \cos^2\theta &= \frac{2}{3} \\
\int_0^\pi d\theta \sin^2\theta \cos^2\theta &= \frac{\pi}{8} \\
\int_0^\pi d\theta \sin^3\theta \cos^2\theta &= \frac{4}{15} \\
\int_0^\pi d\theta \sin^4\theta \cos^2\theta &= \frac{\pi}{16}
\end{aligned}
$$