8.06 SPRING 2003

MIDTERM EXAM SOLUTIONS SET

1. a) T

   b) F (its $2u^2$)

   c) F

   d) F (its α^5)

   e) F

   f) T

   g) F

   h) T

   i) F

   j) F

2 a) $E^0 = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)$

There are three states with $E^0 = \frac{5}{2}\hbar\omega$.

They are: $|100\rangle, |010\rangle, |001\rangle$

b) $H' = Q\, x_1 x_2$

$= Q \frac{\hbar}{2m\omega}(a_1 + a_1^+)(a_2 + a_2^+)$

$= \frac{Q\hbar}{2m\omega}\left(a_1 a_2 + a_1 a_2^+ + a_1^+ a_2 + a_1^+ a_2^+\right)$

c) $H'|100\rangle = \frac{Q\hbar}{2m\omega}|010\rangle + \frac{Q\hbar}{2m\omega}\sqrt{2}\,|210\rangle$

$H'|010\rangle = \frac{Q\hbar}{2m\omega}|100\rangle + \frac{Q\hbar}{2m\omega}\sqrt{2}\,|120\rangle$

$H'|001\rangle = \frac{Q\hbar}{2m\omega}|111\rangle$

$\therefore \langle 010|H'|100\rangle = \langle 100|H'|010\rangle = \frac{Q\hbar}{2m\omega}$

all other matrix elements of H' within the 3×3 subspace vanish. If we order the states in the subspace: $|100\rangle, |010\rangle, |001\rangle$ then in the subspace, $H' = \frac{Q\hbar}{2m\omega}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2 d) We diagonalize H' :

Eigenstates (the "good" states)	Eigenvalue
$\frac{1}{\sqrt{2}}(\|100\rangle + \|010\rangle)$	$+Q\hbar/2m\omega$
$\frac{1}{\sqrt{2}}(\|100\rangle - \|010\rangle)$	$-Q\hbar/2m\omega$
$\|001\rangle$	0

So, the first order corrections are

$$+\frac{Q\hbar}{2m\omega}, \quad -\frac{Q\hbar}{2m\omega}, \quad 0 .$$

e) All three states in the subspace have nonzero matrix elements with states outside the subspace. For example, $\langle 111 | H' | 001 \rangle = \frac{Q\hbar}{2m\omega}$.

$\therefore$ The second order corrections to all three eigenvalues are <u>nonzero</u>.

2 f) $\cancel{\langle 000}$

$$H'|000\rangle = \frac{Q\hbar}{2m\omega}|110\rangle$$

$$\therefore \langle 000|H'|000\rangle = 0$$

$\therefore$ 1st order correction is zero.

$$\text{Second order correction} = \frac{|\langle 110|H'|000\rangle|^2}{E_{000} - E_{110}}$$

$$= \frac{(Q\hbar/2m\omega)^2}{\frac{3}{2}\hbar\omega - \frac{7}{2}\hbar\omega}$$

$$= -\frac{Q^2\hbar^2}{4m^2\omega^2(2\hbar\omega)}$$

$$= -\frac{Q^2\hbar}{8m^2\omega^3}$$

3 a) increases ; increases

b) $\vec{B} = (0, 0, B)$

In order to have $[H, P_x] = 0$, don't want any $\vec{x}$ in ~~$\vec{B}$~~ $\vec{A}$. $\therefore$ choose:

$$\vec{A} = (-By, 0, 0)$$

$$H = \frac{1}{2m}\left(P_x + \frac{qBy}{c}\right)^2 + \frac{P_y^2}{2m}$$

$$= \frac{1}{2m}\left(P_x - \frac{eBy}{c}\right)^2 + \frac{P_y^2}{2m}$$

Define $\omega_L \equiv \frac{eB}{mc}$. Then

$$H = \frac{1}{2} m \omega_L^2 \left(y - \frac{c P_x}{eB}\right)^2 + \frac{P_y^2}{2m}$$

c) Acting on eigenstates of P_x with eigenvalue $\hbar k_x$,

$$H = \frac{1}{2} m \omega_L^2 \left(y - \frac{\hbar c k_x}{eB}\right)^2 + \frac{P_y^2}{2m}$$

Wave functions are $e^{i k_x x} \times$ (harmonic oscillator state in y, centered at $y = \frac{\hbar c k_x}{eB}$)

Center of wave function in y must be between $y = -L/2$ and $y = L/2$.

$$\therefore \ -\frac{L}{2} < \frac{\hbar c k_x}{eB} < \frac{L}{2}$$

$$\therefore \ -\frac{LeB}{2\hbar c} < k_x < \frac{LeB}{2\hbar c}$$

d) Spacing between allowed k_x's is $\frac{2\pi}{L}$

$$\therefore \ \text{Number of allowed } k_x\text{'s} = \frac{LeB/\hbar c}{2\pi/L}$$

$$= \frac{L^2 eB}{2\pi\hbar c} = \frac{L^2 eB}{hc}$$

$$= L^2/A_B \quad \text{with} \quad \boxed{A_B = \frac{hc}{eB}}$$

e) $[h] = g\ cm^2\ s^{-1}$

$[c] = cm\ s^{-1}$

$[eB] = g\ cm\ s^{-2}$

$$\Rightarrow \left[\frac{hc}{eB}\right] = \frac{g\ cm^3 s^{-2}}{g\ cm\ s^{-2}} = cm^2 \ \checkmark\checkmark$$

Note: $\vec{F} = e\frac{\vec{v}}{c}\times\vec{B}$

$\therefore [eB] = [F]$

$[e^2] = [F \times r^2] = g\ cm^3\ s^{-2}$

$[e] = g^{1/2}\ cm^{3/2}\ s^{-1}$

$[B] = g^{1/2}\ cm^{-1/2}\ s^{-1}$

3 f) $\vec{E} = (0, E, 0)$

$\therefore \phi = -Ey$

$$H = \frac{1}{2}m\omega_L^2\left(y - \frac{\hbar c k_x}{eB}\right)^2 + \frac{p_y^2}{2m} + eEy$$

Need to complete the square, inorder to make interpretation easy:

$$H = \frac{p_y^2}{2m} + \frac{1}{2}m\omega_L^2\left(y - \frac{\hbar c k_x}{eB} + \frac{eE}{m\omega_L^2}\right)^2$$
$$+ \frac{\hbar c k_x E}{B} - \frac{e^2E^2}{2m\omega_L^2}$$

Range of allowed k_x's is given by:

$$-\frac{L}{2} < \left(\frac{\hbar c k_x}{eB} - \frac{eE}{m\omega_L^2}\right) < \frac{L}{2}$$

$$\boxed{-\frac{LeB}{2\hbar c} + \frac{e^2EB}{\hbar c m\omega_L^2} < k_x < \frac{LeB}{2\hbar c} + \frac{e^2EB}{\hbar c m\omega_L^2}} \quad *$$

Energy eigenvalues $= (n+\frac{1}{2})\hbar\omega_L + \frac{\hbar c k_x E}{B} - \frac{e^2E^2}{2m\omega_L^2}$

where k_x lies in ~~the~~ the range ⊛

do not need to say this for full credit.

The Landau levels are not degenerate.
Each is broadened into a band of width

$$\frac{\hbar c E}{B} \times \frac{LeB}{\hbar c} = eLE$$

and ~~shifted~~ the center of this band is
shifted by $-\frac{e^2E^2}{2m\omega_L^2} + \frac{e^2 E B}{\hbar c m \omega_L^2}$.

If all values of k_x in the range ⊛ describe occupied states, because the range of allowed k_x's is centered at $+\frac{e^2EB}{\hbar c m\omega_L^2} = +\frac{eE}{\hbar\omega_L}$ there is a "net positive k_x"

$\Rightarrow$ net probability current in $+x$ direction

$\Rightarrow$ net electric current in $-x$ direction.

4) $\psi(x) = N e^{-x^2/2w^2}$

where N fixed by $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$

$$\Rightarrow N^2 \int_{-\infty}^{\infty} dx \, e^{-x^2/w^2} = 1$$

$$\Rightarrow N^2 w \, c_1 = 1$$

where c_1 is a dimensionless constant.

$N = \frac{1}{\sqrt{w c_1}}$ (all c_n's below are dimensionless constants)

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \alpha |x|^3 \right] \psi(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left[+ \frac{\hbar^2}{2m} \left| \frac{\partial \psi}{\partial x} \right|^2 + \alpha |\psi|^2 |x|^3 \right] dx$$

$$= 2N^2 \int_{0}^{\infty} \left[\frac{\hbar^2}{2m} \left(\frac{x}{w^2} \right)^2 e^{-x^2/w^2} + \alpha e^{-x^2/w^2} x^3 \right] dx$$

$$= 2N^2 \frac{\hbar^2}{2m} \frac{1}{w^2} w \underbrace{\int_0^{\infty} y^2 e^{-y^2} dy}_{\equiv c_2} + 2N^2 \alpha w^3 w \underbrace{\int_0^{\infty} y^3 e^{-y^2} dy}_{\equiv c_3}$$

$$= \frac{w}{w c_1} \frac{\hbar^2}{m} \frac{c_2}{w^2} + \cancel{2N^2\alpha} \frac{2w}{w c_1} \alpha w^3 c_3$$

Define $C_4 \equiv \frac{C_2}{C_1}$ $\quad C_5 \equiv \frac{2C_3}{C_1}$

Then

$$\langle H \rangle = \frac{C_4 \hbar^2}{m W^2} + C_5 \alpha W^3$$

$$\frac{\partial \langle H \rangle}{\partial W} = 0 = -\frac{2C_4 \hbar^2}{m W^3} + 3 C_5 \alpha W^2$$

$$\therefore W = \left(\frac{2C_4 \hbar^2}{3 C_5 m \alpha} \right)^{1/5}$$

$\therefore$ characteristic size $\sim \dfrac{\hbar^{2/5}}{m^{1/5} \alpha^{1/5}}$

Ground state energy $\sim \dfrac{\hbar^2}{m W^2} + \alpha W^3$

Both terms go like: $\dfrac{\alpha^{2/5} \hbar^{6/5}}{m^{3/5}}$