Quantum Physics III (8.06) Spring 2005
Solution Set 9

April 29, 2005

1. **Scattering from a Reflectionless Potential (8 points)**

   (a) **(no points)** From the previous problem set, we know that $\psi_0(x) = A\text{sech}(ax)$ is a bound state with energy $E_0 = -\frac{\hbar^2 a^2}{2m}$.

   (b) **(2 points)** We now consider the wave function $\psi(x) = (k/a + i\tanh(ax))e^{ikx}$. Since

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi - 2ak\text{sech}^2(ax)e^{ikx} - 2ia^2e^{ikx}\tanh(ax)\text{sech}^2(ax) = -(k^2 + 2a^2\text{sech}^2(ax))\psi,$$

   we find

$$\begin{aligned} H\psi &= -\frac{\hbar^2}{2m}\left[-k^2\psi - 2a^2\text{sech}^2(ax)\psi\right] - \frac{\hbar^2 a^2}{m}\text{sech}^2(ax)\psi \\ &= \frac{\hbar^2 k^2}{2m}\psi. \end{aligned}$$

   This is true for any (real) k, so we have a continuum of scattering states ψ_k.

   (c) **(4 points)** In order to study scattering, we ask what happens to a plane wave sent in from $x = -\infty$. When the particle interacts with the potential, some of the wave will be transmitted and some reflected, so that asymptotically the solution should reduce to $\psi(x \to -\infty) = e^{ikx} + Re^{-ikx}$ on the left and $\psi(x \to \infty) = Te^{ikx}$ on the right. (Note that Griffiths defines $T_{Griffiths} = |T_{here}|^2$. You should get full credit for using Griffiths definition in this part of the problem, provided it is clear which definition you are using.)

   Now, at a given energy $E = \frac{\hbar^2 k^2}{2m}$, the general solution to the Schrodinger equation for this potential is $\psi_g = A\psi_k + B\psi_{-k}$. In order to study scattering, we ask that as $x \to \infty$, this solution should reduce to an outgoing plane wave. Since $\lim_{x\to\infty}\tanh(ax) = 1$, we find

$$\psi_g(x \to \infty) = A\frac{k + ia}{a}e^{ikx} - B\frac{k - ia}{a}e^{-ikx}.$$

   Therefore, we need to take $B = 0$. Note that this is the *only* boundary conditions that we have to impose. So, we can see immediately that there are *no* terms in our scattering solution proportional to e^{-ikx}, and therefore there is no reflected wave.

   In order to find T, consider the limit $x \to -\infty$ which tells us the coefficient of the incoming wave:

$$\lim_{x\to-\infty}\psi = A\frac{k - ia}{a}e^{ikx},$$

   and therefore

$$T = \frac{k + ia}{k - ia} = 1.$$

   From this expression we can see that $T^* = T^{-1}$, and therefore that $|T| = 1$— which we must have by flux conservation, as there is no reflected wave.

   (d) **(2 points)** As we can see from part (c), $T(k)$ has a pole at $k = ia$. In terms of energy, the location of the pole is at the bound state energy, $E_0 = -\frac{\hbar^2 a^2}{2m}$.

2. **Simple Properties of Cross Sections (14 points)**

(a) **(2 points)** The incident flux is

$$\vec{S}_i = \frac{\hbar k}{m}\hat{z},$$

while the scattered flux is (to leading order in $1/r$)

$$\vec{S}_s = \frac{\hbar k}{m}\frac{|f|^2}{r^2}\hat{r}.$$

(b) **(2 points)** Using part (a),

$$\frac{d\sigma}{d\Omega}d\Omega = \lim_{r\to\infty}\frac{S_s\cdot\hat{r}}{|S_i|}dA = \frac{|f|^2}{r^2}r^2 d\Omega,$$

and therefore $\frac{d\sigma}{d\Omega} = |f|^2$.

(c) **(10 points)** From conservation of probability, we must have $\int \nabla\cdot S_{total} = 0$, and therefore $\int S_{total}\cdot dA = 0$, with the integral is taken over the boundary of space. Now, $S_{total} = S_i + S_s + S_{int}$, where

$$\begin{aligned} S_{int} &= \frac{\hbar}{2mi}\left[e^{-ikz}\nabla\frac{f}{r}e^{ikr} + e^{-ikr}\frac{f^*}{r}\nabla e^{ikz} - c.c.\right] \\ &= \frac{\hbar k}{m}\frac{1}{r}\mathrm{Re}\left[fe^{ik(r-z)}\right](\hat{z}+\hat{r}). \end{aligned}$$

Let's consider $\int S_{total}\cdot dA = 0$ term by term. The first term we consider is $\int S_{int}\cdot dA$. The easiest coordinate system to use is cylindrical coordinates, in which we take the boundary of space to be two planes at $z = \pm\infty$. (You can check that the contribution to the integral from $\rho\to\infty$ is sub leading.) We have, then,

$$\begin{aligned} \int S_{int}\cdot dA &= \frac{\hbar k}{m}\int d\phi\,\rho d\rho\,\frac{1}{r}\mathrm{Re}\left[fe^{ik(r-z)}\right](\hat{z}+\hat{r})\cdot\hat{z} \\ &\simeq \frac{\hbar k}{m}\int d\phi\,\rho d\rho\,\frac{1}{z}\mathrm{Re}\left[f(\theta=0)e^{ik\rho^2/2z}\right](\hat{z}\pm\hat{z})\cdot\hat{z}, \end{aligned}$$

where in the second line I have used the approximations

$$\begin{aligned} \hat{r} &\simeq \pm\hat{z}, \\ r &= |z| + \frac{\rho^2}{2|z|} + \ldots, \\ \theta &\simeq 0. \end{aligned}$$

Because at $z = -\infty$, $\hat{z} = -\hat{r}$, there is no contribution from the plane at $z = -\infty$. The plane at $z = \infty$ contributes

$$\begin{aligned} \int S_{int}\cdot dA &= \frac{4\pi\hbar k}{mz}\int \rho d\rho\,\mathrm{Re}\left[f(\theta=0)e^{ik\rho^2/2z}\right] \\ &= \frac{4\pi\hbar k}{m}\int da\,\mathrm{Re}\left[f(\theta=0)e^{ika}\right] \\ &= \frac{4\pi\hbar k}{m}\lim_{\alpha\to 0}\int da\,e^{-\alpha a}\left[\mathrm{Re}f(\theta=0)\,\cos(ka) - \mathrm{Im}f(\theta=0)\,\sin(ka)\right] \\ &= -\frac{4\pi\hbar}{m}\mathrm{Im}f(\theta=0). \end{aligned}$$

The other two terms are simpler, as $\int \nabla \cdot S_i = 0$ and

$$\int S_s \cdot dA = \frac{\hbar k}{m} \int d\Omega |f|^2.$$

Conservation of flux then tells us

$$\int |f|^2 d\Omega = \frac{4\pi}{k} \text{Im} f(0).$$

3. **Born Approximations for Scattering from Yukawa and Coulomb Potentials (15 points)**

(a) **(6 points)** The Yukawa potential is spherically symmetric, so $f(\theta) = -\frac{2m}{\hbar^2 \kappa} \int_0^\infty dr\, rV(r) \sin \kappa r$, where $\kappa = |\vec{k} - \vec{k}'|$. The integral we need to perform is

$$\begin{aligned} \int dr\, e^{-\mu r} \sin \kappa r &= \frac{1}{2i} \left[\frac{e^{-(\mu + i\kappa)r}}{\mu + i\kappa} - \frac{e^{-(\mu - i\kappa)r}}{\mu - i\kappa} \right] \\ &= \frac{\kappa}{\mu^2 + \kappa^2}. \end{aligned}$$

Using this,

$$f(\theta) = -\frac{2m\beta}{\hbar^2(\mu^2 + \kappa^2)}.$$

From this we find

$$\frac{d\sigma}{d\Omega} = \left(\frac{2m\beta}{\hbar^2(\mu^2 + \kappa^2)} \right)^2.$$

The total cross-section is

$$\sigma = \int |f|^2 d\Omega = \left(\frac{2m\beta}{\hbar^2} \right)^2 \int \sin \theta d\theta d\phi \frac{1}{(\mu^2 + 4k^2 \sin^2 \frac{\theta}{2})^2}.$$

To evaluate the θ integral, make the variable change $a = 4k^2 \sin^2(\theta/2)/\mu^2$. Then the θ integral becomes $\frac{1}{2\mu^2 k^2} \int_0^{4k^2/\mu^2} \frac{da}{(1+a)^2} = \frac{2}{\mu^2(\mu^2 + 4k^2)}$, and

$$\sigma = \frac{4\pi}{\mu^2 + 4k^2} \left(\frac{2m\beta}{\hbar^2 \mu} \right)^2 = \frac{4\pi}{\hbar^2 \mu^2 + 8mE} \left(\frac{2m\beta}{\hbar \mu} \right)^2.$$

(b) **(2 points)** Putting $\beta = Q_1 Q_2$ and $\mu = 0$, we find

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mQ_1 Q_2}{2\hbar^2 k^2 (1 - \cos \theta)} \right)^2 = \left(\frac{Q_1 Q_2}{4E} \right)^2 \frac{1}{\sin^4 \theta/2},$$

which is precisely the Rutherford cross-section.

(c) **(2 points)** To prevent confusion, I will use the symbol T for thickness. Since the number of particles scattered per unit solid angle per unit time per scatterer is $\frac{d\sigma}{d\Omega} \frac{d^2 N}{dt dA}$, and the number density of scatterers is n, the number of particles scattered into unit solid angle per unit time is $\int dA\, T n \frac{d\sigma}{d\Omega} \frac{d^2 N}{dt dA} = \frac{d\sigma}{d\Omega} \frac{dN}{dt} nT$. This is independent of beam area and beam uniformity, as desired. The key point here is that the factors of area cancel *independently* for each area element, so non uniformity of the beam is not an issue.

(d) **(3 points)** With the numbers

$$Q_1 = 2e \qquad Q_2 = 79e \qquad \theta = \frac{\pi}{2}$$
$$T = 10^{-6}m \qquad d\Omega = 10^{-4}\text{rads}^2 \qquad E = 8 \ \text{MeV}$$
$$n = \rho_{Au}/m_{Au} = 5.9 \times 10^{28}\text{atoms}/m^3,$$

we find that the number of scattered alpha particles seen in the detector per second is 3.7.

(e) **(2 points)** We now need to take the factor of $(\sin\theta/2)^{-4}$ in the Rutherford cross-section into consideration. At the 4 angles we are asked to consider, $\sin^4\theta/2$ takes on the following values:

$$\begin{array}{rl} 5.8 \times 10^{-5} & \theta = 10^o \\ 2.1 \times 10^{-2} & \theta = 45^o \\ .73 & \theta = 135^o \\ .98 & \theta = 170^o. \end{array}$$

In part (d), when we took $\theta = 90^o$ and therefore $\sin^4\theta/2 = .25$, we found that there were 3.7 particles per second scattered into the detector. Using this, the number of particles observed per second in the detector at the various angular locations are

$$\begin{array}{rl} 1.6 \times 10^{4} & \theta = 10^o \\ 44 & \theta = 45^o \\ 1.3 & \theta = 135^o \\ .94 & \theta = 170^o. \end{array}$$

Note that the observed number of particles shoots up very sharply near $\theta = 0$.

4. **The Born Approximation in One Dimension (15 points)**

(a) **(6 points)** We need to show that $\psi(x) = \psi_0(x) - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} e^{ik|x-x_0|}V(x_0)\psi(x_0)dx_0$ satisfies the Schrodinger equation with eigenvalue $\frac{\hbar^2 k^2}{2m}$. Now,

$$\frac{\partial^2}{\partial x^2}e^{ik|x-y|} = \left[2ik\delta(x-y) - k^2\right]e^{ik|x-y|},$$

so

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = \frac{\hbar^2 k^2}{2m}\left(\psi_0 - \frac{im}{\hbar^2 k^2}\int dy\, e^{ik|x-y|}V(y)\psi(y)\right) - \psi(x)V(x).$$

It is therefore easy to see that

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x) = \frac{\hbar^2 k^2}{2m}\psi(x).$$

(b) **(5 points)** Using $\psi_0(x) \simeq Ae^{ikx}$, the first Born approximation becomes

$$\psi(x) \simeq Ae^{ikx} - \frac{Aim}{\hbar^2 k}\int dy\, e^{ik|x-y|}V(y)e^{iky}.$$

We want to find the reflection coefficient, which as Griffiths defines it is $R = |\text{reflected wave}|^2/|\text{incident wave}|^2$ (since this problem comes from Griffiths, we will use his definitions for R and T in this problem). The reflected wave is the component of ψ that behaves like e^{-ikx} at $x \to -\infty$. Now, as $x \to -\infty$, $|x-y| = y-x$, and thus the coefficient of the reflected wave is $-\frac{Aim}{\hbar^2 k}\int dy\, e^{2iky}V(y)$. The incident wave is the component of ψ that behaves like e^{ikx} at $x \to -\infty$, and the coefficient of this piece of ψ is A. Therefore,

$$R = \left|\frac{m}{\hbar^2 k}\int dy\, e^{2iky}V(y)\right|^2. \tag{1}$$

(c) **(4 points)** First, we set $V(x) = -\alpha\delta(x)$. In this case, the integral in equation (1) becomes $-\int dy\, e^{2iky}\alpha\delta(y) = -\alpha$. The reflection coefficient is therefore $R = \frac{m^2\alpha^2}{\hbar^4 k^2} = \frac{m\alpha^2}{2\hbar^2 E}$. From this, we find the first Born approximation to the transmission coefficient,

$$T = 1 - R = 1 - \frac{m\alpha^2}{2\hbar^2 E}.$$

The exact answers are, defining $w = \frac{m\alpha^2}{2\hbar^2 E}$,

$$R_{exact} = \frac{w}{w+1}, \qquad T_{exact} = \frac{1}{1+w},$$

so the first Born approximation has found the first term in an expansion of R_{exact}, T_{exact} in small w. Small w means that the strength of the potential (measured by α) is small compared to the energy of the particle, and so this means that the Born approximation is good in the weak scattering regime.

Second, we set $V(x) = -V_0$ when $|x| < a$ and 0 otherwise. In this case, the integral of (1) is $-V_0\int_{-a}^{a} dy\, e^{2iky} = -\frac{V_0\sin(2ka)}{k}$. The reflection coefficient is then $R = \left(\frac{m}{\hbar^2 k^2}V_0\sin(2ka)\right)^2$, and the transmission coefficient is

$$T = 1 - R = 1 - \left(\frac{V_0}{2E}\sin(2ka)\right)^2.$$

The exact answer is

$$T_{exact} = \left(1 + \frac{V_0^2}{4E(E+V_0)}\sin^2(2ka)\right)^{-1} \simeq 1 - \left(\frac{V_0}{2E}\sin(2ka)\right)^2$$

where the approximation holds when $V_0/E \ll 1$, that is, in the weak scattering regime. Again, we have found the first term in the weak scattering expansion of the exact answer.

5. **The Size of Nuclei (8 points)**

(a) **(3 points)** Using the charge distribution $\rho(r) = \frac{3Z}{4\pi R^3}$ for $r \le R$ and 0 otherwise, we find

$$\int r^2 dr\, \sin\theta d\theta\, d\phi\, e^{iqr\cos\theta}\rho(r) = \frac{3Z}{R^3 q^2}\left[\frac{1}{q}\sin qR - R\cos qR\right],$$

and therefore

$$f = \frac{6mZe^2}{\hbar^2 q^2 (qR)^3}(\sin qR - qR\cos qR).$$

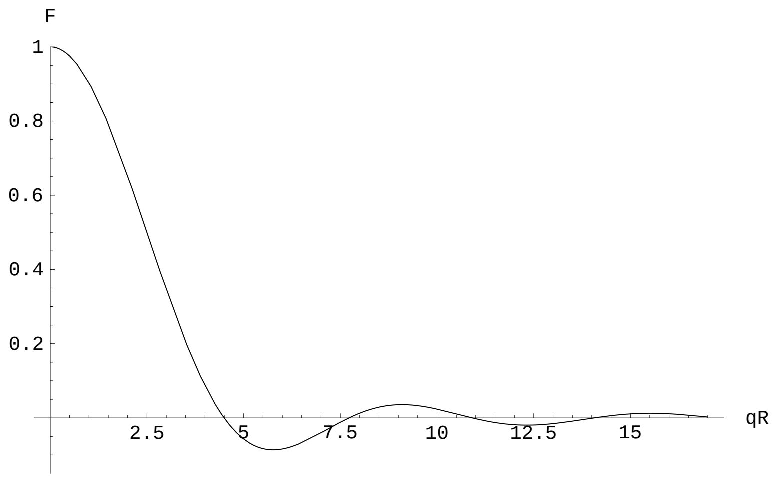

Figure 1: The form factor $F(qR)$.

The scattering cross-section is

$$\frac{d\sigma}{d\Omega} = |f|^2 = \left(\frac{6mZe^2}{\hbar^2 q^2 (qR)^3}\right)^2 (\sin qR - qR\cos qR)^2.$$

(b) **(2 points)** For a point particle, $\rho(x) = Z\delta^3(r)$, so the scattering amplitude is $f_0 = 2me^2 Z/(\hbar^2 q^2)$. We define the form factor

$$F(qR) = \frac{f}{f_0} = \frac{3}{(qR)^3}(\sin qR - qR\cos qR).$$

We plot this function in figure 1.

(c) **(3 points)** As θ goes from 0 to π, q goes from 0 to $2E/(\hbar c)$. In the relativistic regime, $qR = \frac{2ER}{\hbar c}\sin\theta/2$, so if $E/(\hbar c) \ll 1/R$, then $qR \ll 1$. In this regime,

$$\begin{aligned} F(qR) &= \frac{3}{(qR)^3}\left(qR - \frac{(qR)^3}{6} + \ldots - \left(qR - \frac{(qR)^3}{2} + \ldots\right)\right) \\ &\simeq 1. \end{aligned}$$

Since the form factor is unity, this means that in this regime there is no difference between the scattering from a point particle and an object of size $R \ll \hbar c/E$.

If $R = (2-7)\times 10^{-15}m$, then we need $E \simeq \hbar c/(2-7)\times 10^{-15}m = 30 - 100$ MeV in order to resolve the finite size of the nucleus.