Quantum Physics III (8.06) Spring 2004

FINAL EXAMINATION

Wednesday May 19, 1:30 pm

You have 3 hours.

There are 9 problems, totalling 180 points. Do all problems.

Answer all problems in the blue books provided.
Write YOUR NAME on EACH blue book you use.
Budget your time wisely, using the point values as a guide. Note also that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

- Conservation of Probability

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \vec{\nabla}\cdot\vec{J}(\vec{r},t) = 0$$

where

$$\rho(\vec{r},t) = |\psi(\vec{r},t)|^2 \; ; \quad \vec{J}(\vec{r},t) = \frac{\hbar}{2im}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

- Operators for Spin-1/2 particle

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$[\hat{x},\hat{p}] = i\hbar \; .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right) \; , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) \; ,$$

and satisfy

$$[\hat{a},\hat{a}^\dagger] = 1 \; .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \; , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \; .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \; .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

- Spherical Coordinates

$$x = r\sin\theta\cos\phi \; ; \qquad y = r\sin\theta\sin\phi \; ; \qquad z = r\cos\theta$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

- Angular Momentum

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k \; ; \quad [\hat{L}^2, \hat{L}_i] = 0$$

$$\hat{L}^2|\ell, m\rangle = \hbar^2\ell(\ell+1)|\ell, m\rangle \; ; \qquad \hat{L}_z|\ell, m\rangle = \hbar m|\ell, m\rangle$$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$[\hat{L}_\pm, \hat{L}^2] = 0 \; ; \qquad [\hat{L}_+, \hat{L}_-] = 2\hbar\hat{L}_z$$

$$\hat{L}^2 = \hat{L}_+\hat{L}_- + L_z^2 - \hbar L_z$$

$$\hat{L}_\pm|\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}|\ell, m\pm 1\rangle$$

- Angular momentum operators in spherical coordinates

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \; ; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle\theta,\phi|\ell, m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \; ; \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi) \; ; \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi) \; ; \qquad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi) \; ;$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

- Hydrogen atom

  Energy eigenvalues are $E_n = -E_I/n^2$ where n, the principal quantum number, is a positive integer, and $E_I = me^4/2\hbar^2 \sim 13.6$ eV.

  Bohr radius is $a_0 = \hbar^2/me^2$

  Wave functions for states which are eigenstates of H, L^2 and L_z have the form $\psi_{n,\ell,m}(\vec{r}) = \frac{1}{r}u_{n,\ell}(r)Y_{\ell,m}(\theta,\phi)$, where

$$u_{n=1,\ell=0}(r) = \frac{2r}{a_0^{3/2}}\exp(-r/a_0)$$

$$u_{n=2,\ell=0}(r) = \frac{2r}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{n=2,\ell=1}(r) = \frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

- Some useful constants:

$$\hbar c = 197 \times 10^{-7} \text{ eV cm}$$

  The mass of the electron is $m_e = 0.511$ MeV/c^2. If B is 1 gauss, then the force eB is 300 eV/cm.

- Particle in an Electric and/or Magnetic Field:

  The Hamiltonian for a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and an electric field $\vec{E} = -\vec{\nabla}\phi$ is:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \tag{1}$$

- Gauge invariance:

  If $\psi(\vec{x},t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x},t) = \exp\left(\frac{iq}{\hbar c}f(\vec{x},t)\right)\psi(\vec{x},t)$$

  solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla}f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

  Suppose that
$$H = H^0 + H'$$
  where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0:
$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \ .$$
  Then, the eigenvalues and eigenstates of the full Hamiltonian H are:
$$E_n = E_n^0 + H'_{nn} + \sum_{m\neq n} \frac{|H'_{nm}|^2}{E_n^0 - E_m^0} + \ldots$$
$$|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m\neq n} \frac{H'_{mn}}{E_n^0 - E_m^0}|\psi_m^0\rangle + \ldots$$
  where $H'_{nm} \equiv \langle\psi_n^0|H'|\psi_m^0\rangle$.

- Connection Formulae for WKB Wave Functions:

  At a turning point at $x = a$ at which the classically forbidden region is at $x > a$:
$$\begin{array}{ccc} \frac{2}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_x^a p(x')dx' - \frac{\pi}{4}\right] & \leftarrow & \frac{1}{\sqrt{\kappa(x)}}\exp\left[-\frac{1}{\hbar}\int_a^x \kappa(x')dx'\right] \\ \frac{1}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_x^a p(x')dx' + \frac{\pi}{4}\right] & \rightarrow & \frac{1}{\sqrt{\kappa(x)}}\exp\left[+\frac{1}{\hbar}\int_a^x \kappa(x')dx'\right] \end{array}$$
  At a turning point at $x = b$ at which the classically forbidden region is at $x < b$:
$$\begin{array}{ccc} \frac{1}{\sqrt{\kappa(x)}}\exp\left[-\frac{1}{\hbar}\int_x^b \kappa(x')dx'\right] & \rightarrow & \frac{2}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_b^x p(x')dx' - \frac{\pi}{4}\right] \\ \frac{1}{\sqrt{\kappa(x)}}\exp\left[+\frac{1}{\hbar}\int_x^b \kappa(x')dx'\right] & \leftarrow & \frac{1}{\sqrt{p(x)}}\cos\left[\frac{1}{\hbar}\int_b^x p(x')dx' + \frac{\pi}{4}\right] \end{array}$$

- Born Approximation to Scattering Amplitude:

$$f(\theta,\phi) = f(\vec{q}) = -\frac{m}{2\pi\hbar^2}\int d^3r \exp(-i\vec{q}\cdot\vec{r})V(\vec{r})$$

where $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transfer.

- Partial Wave Analysis of Scattering from a spherically symmetric potential:

$$f(\theta) = \sum_l (2\ell+1)P_l(\cos\theta)\frac{\exp(2i\delta_\ell)-1}{2ik}$$

$$\sigma = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\sin^2\delta_\ell$$

- Legendre Polynomials

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z \\ P_2(z) &= \frac{3z^2-1}{2} \\ P_3(z) &= \frac{5z^3-z}{2} \end{aligned}$$

- Time Dependent Perturbation Theory:

  Consider a system with the Hamiltonian

$$H = H_0 + V(\vec{r})\cos(\omega t)\exp(-t^2/T^2)$$

and denote the matrix element of V between eigenstates of H_0 named $|a\rangle$ and $|b\rangle$ by V_{ab}. Then, in the large T limit, if the system is initially in state $|a\rangle$, the probability that it is in the state $|b\rangle$ for $t \to +\infty$ is:

$$P_{a\to b} = \frac{\pi}{2\hbar^2}|V_{ab}|^2\delta(\omega-\omega_{ab})\sqrt{\pi}T$$

where $\hbar\omega_{ab} = E_b - E_a$.

1. **Which of these states are degenerate? (15 points)**

   Consider the $n = 2$ states of hydrogen. Do *not* ignore spin in this problem.

   (a) (5 points) What are the quantum numbers of the $n = 2$ eigenstates of the Hamiltonian, including fine structure? Specify which states are degenerate.

   (b) (2 points) Now including both the fine structure and the Lamb shift, which of these states are degenerate?

   (c) (3 points) Turn on a weak magnetic field. Including fine structure effects, but neglecting the Lamb shift, which states are degenerate?

   (d) (5 points) Now make the magnetic field very strong, ignore fine structure and the Lamb shift, and assume the magnetic field dependent term in the Hamiltonian is proportional to $\vec{B} \cdot (\vec{L} + 2\vec{S})$. What are the quantum numbers of the $n = 2$ eigenstates of the Hamiltonian? Specify which states are degenerate.

2. **Fermions in a Box (15 points)**

   Consider the ground state of N noninteracting nonrelativistic spin-1/2 fermions with mass m in a $L \times L \times L$ cubic box. Assume the wave functions satisfy periodic boundary conditions. Assume that N is very large – think of it as of order 10^{23} if you like. What is the ground state energy of the N fermions? (Your answer should depend on N, L, m and constants of nature.)

3. **Exciting a Hydrogen Atom (10 points)**

   Ignore spin in this problem.

   A hydrogen atom initially (i.e. at $t \to -\infty$) in its ground state is exposed to a pulse of ultraviolet light that can be thought of as a time varying electric field which points precisely in the z-direction at all times and whose magnitude is given by

   $$|\vec{E}(t)| = E_0 \cos(\omega t) \exp(-t^2/T^2) \ ,$$

   with E_0, ω and T all constants. The frequency ω satisfies

   $$\hbar\omega = \frac{8}{9}(13.6 \text{ eV}) \ .$$

   The constant T is large compared to all other timescales in the problem.

   At $t \to +\infty$, the atom is, in general, in some superposition of energy eigenstates. Working to first order in time-dependent perturbation theory, what are all the state(s) that could arise in this superposition? (Specify the state(s) by their quantum numbers n, ℓ and m.)

4. **Perturbing a Three-Dimensional Harmonic Oscillator (18 points)**

A particle of mass m moves in a three dimensional harmonic oscillator, with Hamiltonian

$$H^0 = \frac{\vec{p}^2}{2m} + \frac{m\omega^2\vec{r}^2}{2} .$$

You know that the energy eigenstates can be labelled by the occupation numbers in the x, y, and z directions:

$$|n_x, n_y, n_z\rangle = \frac{\left[a_x^\dagger\right]^{n_x} \left[a_y^\dagger\right]^{n_y} \left[a_z^\dagger\right]^{n_z}}{\sqrt{n_x! n_y! n_z!}} |0,0,0\rangle .$$

Suppose the oscillator is perturbed by adding an interaction of the form

$$H' = \lambda xyz$$

where λ is a small constant.

(a) (10 points) What is the energy of the state $|n_x, n_y, n_z\rangle$ to first order in λ? Make sure to justify your answer carefully.

(b) (8 points) What is the energy of the ground state to second order in λ?

5. **A Variational Problem (22 points)**

Consider a particle with mass m moving in the one-dimensional potential

$$V(x) = \lambda x^4 ,$$

with λ a positive constant.

(a) (18 points) Consider a single-parameter ansatz for the wave function consisting of ground-state wave functions for a simple harmonic oscillator with frequency ω, where ω is the variational parameter. Find the value of ω that minimizes $\langle H \rangle$ and obtain an upper bound on the ground state energy.

(b) (4 points) Write down a one-parameter ansatz that you could use, with the variational principle, to obtain an upper bound on the energy of **the first excited state.** Explain the reasoning behind your choice of ansatz, but do not go farther with the calculation than writing down an ansatz.

6. **A Time-Dependent Two-State System (15 points)**

Consider a two-state system with basis vectors $|1\rangle$, $|2\rangle$. In this basis the Hamiltonian is

$$E_0 \begin{pmatrix} \cos\left(\frac{\pi t}{2T}\right) & \sin\left(\frac{\pi t}{2T}\right) \\ \sin\left(\frac{\pi t}{2T}\right) & -\cos\left(\frac{\pi t}{2T}\right) \end{pmatrix} .$$

E_0 and T are constants.

At time $t = 0$, the state of the system is $|1\rangle$.

Denote the state at time t by $|\psi(t)\rangle$

(a) (3 points) What criterion must T satisfy in order for the time evolution in this system to be well-approximated as adiabatic?

(b) (6 points) In parts (b) and (c), assume that the criterion of part (a) is satisfied. What is $|\langle\psi(T)|1\rangle|^2$?

(c) (6 points) What is $|\psi(4T)\rangle$? Make sure to include the correct overall phase. [Note: you do not have time to derive the overall phase from scratch; this part of the problem tests whether you understand and remember enough to come up with it. If you do not, move on.]

7. **Semiclassical Approximation (25 points)**

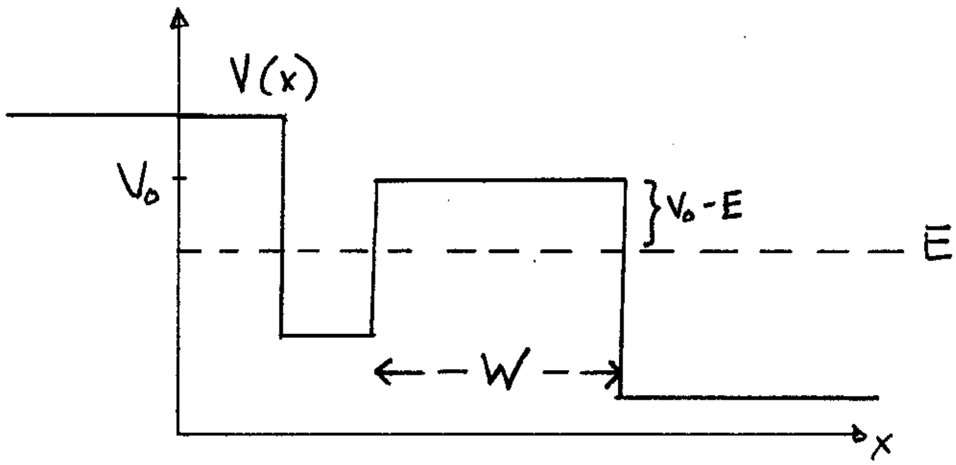
(a) (6 points) Write down the semiclassical wave function for a particle in a classically allowed region, and explain how it incorporates the classical notion that the particle spends more time in regions where it moves slowly.

(b) (10 points) Consider a particle with mass m moving in one dimension in the potential

$$V(x) = \lambda x^4,$$

where λ is a positive constant. Derive an expression that determines the bound state energies in this potential in the semi-classical approximation. [Your expression will include an integral; do NOT attempt to evaluate the integral.]

(c) (9 points) A particle of mass m is trapped in a "quasibound state", meaning that classically it is bound but quantum mechanically it can escape by tunnelling. The potential is sketched here, with the energy E of the quasibound state indicated on the sketch:



You have done a calculation and found that the probability the particle tunnels out of its quasibound state during the coming year is 10^{-10}. The result of your calculation includes a prefactor, and an exponential term. In the following, you may ignore changes in the prefactor.

i. Suppose the mass of the particle were $2m$ instead of m. What would the probability that the particle tunnels out of its quasibound state during the coming year be?

ii. Suppose the width of the barrier, shown in the sketch, were $2W$ instead of W. What would the probability that the particle tunnels out of its quasibound state during the coming year be?

iii. Suppose the height of the barrier, $V_0 - E$, were doubled. What would the probability that the particle tunnels out of its quasibound state during the coming year be?

8. **Electron in Crossed Electric and Magnetic Field (30 points)**

Consider an electron in a uniform magnetic field $\vec{B} = (0, 0, B_0)$ and a uniform electric field $\vec{E} = (0, E_0, 0)$. Assume that the particle is restricted to move in the xy-plane.

(a) (5 points) Write the Hamiltonian in a gauge in which p_x commutes with H.

(b) (15 points) Let $\omega_L = \frac{qB_0}{mc}$ (the Larmor frequency) and $v_D = \frac{cE_0}{B_0}$ (the classical drift velocity). Show that for any value of a and any $n = 0, 1, 2, \ldots$ there is an energy eigenstate with energy $\frac{1}{2}mv_D^2 - qEa + (n + \frac{1}{2})\hbar\omega_L$. Relate a to a property of the wave functions of the energy eigenstates.

(c) (5 points) Suppose now that, instead of being free to move anywhere within an infinite plane, the electron is required to stay within a bounded extent in y, say $-L/2 < y < L/2$. Does this restrict the allowed values of a or n? If so, how?

(d) (5 points) Suppose now that the electron is required to stay within a bounded extent in x, say $-W/2 < x < W/2$. Does this restrict the allowed values of a or n? If so, how? A qualitative answer will suffice.

9. **Scattering From a δ-Function Shell (30 points)**

Consider s-wave ($\ell = 0$) scattering of particles with mass m from the potential

$$V(r) = \lambda \frac{\hbar^2}{2mb} \delta(r - b) \ .$$

In this problem, we will only investigate very low energy scattering, fully specified by the scattering length.

Recall that at large r the s-wave radial wave function is

$$R(r) \propto \frac{1}{r} u(r)$$

with

$$u(r) = \sin(kr - \delta_0) \ .$$

And, recall that the scattering length a is defined by

$$\lim_{k \to 0} \frac{\tan \delta_0}{k} = -a \ .$$

(a) (6 points) What is the scattering amplitude f and the total cross-section σ for low energy scattering? Your answers may depend on a and/or k, but should not depend on parameters of the potential (except via the fact that a does).

(b) (6 points) Use the Born approximation to evaluate the scattering amplitude and the total cross-section, and from that deduce the scattering length a, assuming that $|\lambda|$ is small.

(c) (6 points) In parts (c), (d) and (e), do not assume that $|\lambda|$ is small. Before doing parts (c) and (d), make sure you read through the entire rest of the problem, including the note at the bottom.

Suppose $\lambda < 0$. Sketch $u(r)$ at very small k for several values of $|\lambda|$, beginning with small $|\lambda|$ and increasing. Label a on each of your sketches. If there are "special" values of $|\lambda|$ for which a is either infinite or zero, make sure to note this, and provide sketches for such cases. You need not determine the numerical value of any "special $|\lambda|$'s".

(d) (6 points) Suppose $\lambda > 0$. Sketch $u(r)$ at very small k for several values of $|\lambda|$, beginning with small $|\lambda|$ and increasing. Label a on each of your sketches. If there are "special" values of $|\lambda|$ for which a is either infinite or zero, make sure to note this, and provide sketches for such cases. You need not determine the numerical value of any "special $|\lambda|$'s".

(e) (6 points) Use the results of your graphical analyses in parts (c) and (d) to make a plot showing $a(\lambda)$ for $-\infty < \lambda < \infty$. Your plot of $a(\lambda)$ should have the correct behavior for $|\lambda|$ near zero, for $\lambda \to -\infty$, and for $\lambda \to +\infty$, and should be labelled clearly enough so that a grader can see that you understand all these limits. Your plot should also highlight the existence of any "special" values of λ.

[Note: In both parts (c) and (d) I have asked you to use graphical methods to determine the qualitative dependence of a on the parameter λ. If you want to, you may instead solve the problem analytically in full, obtaining an analytical expression for $a(\lambda)$. If you do this correctly, you will get full credit for parts (c) and (d). Your plot in part (e) should then be based on your analytical solution, and instead of just indicating the existence of special values of λ it must give their numerical values.]