8.06 SPRING 2004

FINAL EXAM

SOLUTION SET

1a) when fine structure is included, CSCO is $\{H, L^2, J^2, J_z\}$ and states specified as $|n, l, j, m_j\rangle$

The $n=2$ states are:

$n\ l\ j\ m_j$	(a)	(b)
$\vert 2\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$	degenerate	degenerate
$\vert 2\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$		
$\vert 2\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$		degenerate
$\vert 2\ 1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$		
$\vert 2\ 1\ \tfrac{3}{2}\ \tfrac{3}{2}\rangle$	degenerate.	degenerate
$\vert 2\ 1\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle$		
$\vert 2\ 1\ \tfrac{3}{2}\ -\tfrac{1}{2}\rangle$		
$\vert 2\ 1\ \tfrac{3}{2}\ -\tfrac{3}{2}\rangle$		

b) Including Lamb shift, the degeneracies are

c) **None** are degenerate, when $\vec{B} \neq 0$.

1 d) CSCO is now $\{H, L^2, L_z, S_z\}$ so

States specified as $|n, l, m_l, m_s\rangle$

$\lvert n \quad l \quad m_l \quad m_s\rangle$	$\underline{m_l + 2m_s}$	
$\lvert 2 \quad 1 \quad 1 \quad \tfrac{1}{2}\rangle$	2	
$\lvert 2 \quad 0 \quad 0 \quad \tfrac{1}{2}\rangle$	1	degenerate
$\lvert 2 \quad 1 \quad 0 \quad \tfrac{1}{2}\rangle$	1	
$\lvert 2 \quad 1 \quad 1 \quad -\tfrac{1}{2}\rangle$	0	degenerate
$\lvert 2 \quad 1 \quad -1 \quad \tfrac{1}{2}\rangle$	0	
$\lvert 2 \quad 1 \quad 0 \quad -\tfrac{1}{2}\rangle$	-1	degenerate
$\lvert 2 \quad 0 \quad 0 \quad -\tfrac{1}{2}\rangle$	-1	
$\lvert 2 \quad 1 \quad -1 \quad -\tfrac{1}{2}\rangle$	-2	

2.

$$N = \overset{\text{spin}}{2} \cdot 4\pi \int_0^{k_F} k^2\, dk \underbrace{\left(\frac{L}{2\pi}\right)^3}$$

density of states in k-space, due to periodic boundary conditions on an $L \times L \times L$ box.

not required for full credit → [Wave functions $\sim e^{ik_x x} e^{ik_y y} e^{ik_z z}$

and $\psi(x+L) = \psi(x)$

$\psi(y+L) = \psi(y)$

$\psi(z+L) = \psi(z)$

$\Rightarrow$ allowed k's spaced by $\frac{2\pi}{L}$]

$$\therefore \quad N = \frac{L^3}{\pi^2}\frac{k_F^3}{3} \Rightarrow k_F^3 = \frac{3\pi^2 N}{L^3}$$ ← 7 points

$$E = 2 \cdot 4\pi \int_0^{k_F} k^2\, dk \left(\frac{L}{2\pi}\right)^3 \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2 L^3}{2\pi^2 m}\frac{k_F^5}{5} = \frac{\hbar^2 L^3}{10\pi^2 m}\left(\frac{3\pi^2 N}{L^3}\right)^{5/3}$$

$$= \frac{(3\pi^2)^{5/3}}{10\pi^2}\frac{\hbar^2 N^{5/3}}{m L^2}$$

3. $\hbar\omega = \frac{8}{9}(13.6\,eV)$ means the atom can be excited only to $n=3$ states.

$\left(E_{n=3} - E_{n=1} = -\frac{1}{9}(13.6eV) - (-13.6eV) = \frac{8}{9}(13.6eV) \right)$

We now ask which matrix elements

$$\langle \underbrace{3\ \ l\ \ m_l}_{\text{excited state}} | z | \underbrace{1\ 0\ 0}_{\text{ground state}} \rangle$$

$n=3$ operator in $H'(t)$

$$H'(t) = -q\, z\, E_0 \cos\omega t\, e^{-t^2/\tau^2}$$

are nonzero.

zero for $l=0,2$. (Parity)

zero for $m_l \neq 0$ (azimuthal integral)

$\therefore$ only $\langle 3\ 1\ 0 | z | 1\ 0\ 0 \rangle$ is nonzero.

So, there is some probability atom is excited to $|310\rangle$ state. [8 points to here]

So, at end of process atom is in some superposition of $|100\rangle$ and $|310\rangle$ states.

4 a) To zeroth order,

$$E^0 = (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega$$

So, this is a problem of degenerate perturbation theory. Must diagonalize $H' = \lambda xyz$ in all subspaces of state with same values of $(n_x + n_y + n_z)$.

$$H' = \lambda xyz = \lambda \left(\frac{\hbar}{2m\omega}\right)^{3/2} (a_x + a_x^+)(a_y + a_y^+)(a_z + a_z^+)$$

From this, we see that $H'|n_x, n_y, n_z\rangle$ is a state whose $n_x + n_y + n_z$ has changed by ± 1 or ± 3.

$\therefore$ $H'_{ab} = 0$ for a, b in a degenerate subspace.

Since all H'_{ab} are zero " " " ",

including the off diagonal, there is no first order correction to the energy

$$E^0 + E^1 = E^0 = (n_x + n_y + n_z + \tfrac{3}{2})\hbar\omega$$

4b)

$$E^{2}_{0,0,0} = \sum_{(n_x,n_y,n_z)\neq(0,0,0)} \frac{|\langle n_x,n_y,n_z|\lambda xyz|0,0,0\rangle|^2}{E_{0,0,0}-E_{n_x n_y n_z}}$$

$$\lambda xyz|0,0,0\rangle = \lambda\left(\frac{\hbar}{2m\omega}\right)^{3/2}|1,1,1\rangle$$

$$\therefore\ E^{2}_{0,0,0} = \frac{\left(\lambda\left(\frac{\hbar}{2m\omega}\right)^{3/2}\right)^2}{E_{0,0,0}-E_{1,1,1}}$$

$$= \frac{\lambda^2\left(\frac{\hbar}{2m\omega}\right)^3}{\frac{3}{2}\hbar\omega-\frac{9}{2}\hbar\omega} = -\frac{\lambda^2}{3\hbar\omega}\left(\frac{\hbar}{2m\omega}\right)^3$$

$$= -\frac{\lambda^2\hbar^2}{24m^3\omega^4}$$

$$\therefore\ E_{0,0,0} = \frac{3}{2}\hbar\omega + 0 - \frac{\lambda^2\hbar^2}{24m^3\omega^4}$$

5. (a) $$H = \frac{p^2}{2m} + \lambda x^4$$

I need to evaluate $\langle 0 | H | 0 \rangle$ where $|0\rangle$ is the g.s. wave fn of a S.H.O. with frequency ω.

$$\langle 0 | x^4 | 0 \rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0 | (a+a^\dagger)(a+a^\dagger)(a+a^\dagger)(a+a^\dagger) | 0 \rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0 | a a a^\dagger a^\dagger | 0 \rangle + \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0 | a a^\dagger a a^\dagger | 0 \rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 (2 + 1)$$

$$= \frac{3}{4} \left(\frac{\hbar^2}{m^2\omega^2}\right)$$

$$\langle 0 | p^2 | 0 \rangle = -\left(\frac{m\omega\hbar}{2}\right) \langle 0 | (a - a^\dagger)(a - a^\dagger) | 0 \rangle$$

$$= -\left(\frac{m\omega\hbar}{2}\right)(-1) \langle 0 | a a^\dagger | 0 \rangle$$

$$= \frac{m\omega\hbar}{2}$$

$$\therefore \langle H \rangle = \frac{\hbar\omega}{4} + \frac{3}{4} \frac{\lambda \hbar^2}{m^2\omega^2}$$ ← 12 points

$$\frac{\partial \langle H \rangle}{\partial \omega} = \frac{\hbar}{4} - \frac{3}{2} \frac{\lambda \hbar^2}{m^2 \omega^3} = 0 \Rightarrow \omega^3 = \frac{6\lambda\hbar}{m^2}$$ ← 4 points

$$\therefore \text{g.s. energy is} \leq \left[\frac{\hbar}{4}\left(\frac{6\lambda\hbar}{m^2}\right)^{1/3} + \frac{3}{4}\frac{\lambda\hbar^4}{m^2}\left(\frac{m^2}{6\lambda\hbar}\right)^{2/3} \right]$$

← 2 points

5 b) Any normalizable one parameter family of ODD wave functions will do.

If the trial wave functions are odd, they have ~~small~~ zero overlap with the true ground state, and in this case the variational method gives a bound on the 1st excited state energy.

Examples: $|1\rangle$, where this is the first excited state of a SHO with frequency ω.

or, $x e^{-ax^2}$ with a the parameter.

or, ...

6. a) At <u>all</u> times, the eigenvalues are

$$E_+ = +E_0$$
$$E_- = -E_0$$

So, criterion is $T \gg \frac{\hbar}{2E_0}$

b) At $t=0$, $H = E_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and the state is $|1\rangle$, which has energy E_+.

At $t=T$, $H = E_0\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. By the adiabatic theorem, the state of the system is ~~that~~ the energy eigenstate with eigenvalue E_+. That is $\left(\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle\right)e^{i\,\text{phase}}$.

$$\therefore \quad |\langle\psi(T)|1\rangle|^2 = 1/2$$

c) At $t=4T$, $H = E_0\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

$$\therefore \quad |\psi(4T)\rangle = \underbrace{e^{-iE_+4T/\hbar}}_{\substack{\text{dynamical}\\ \text{phase}}}\ \underbrace{e^{i\gamma}}_{\substack{\text{Berry's}\\ \text{phase}}}|1\rangle$$

6 c) cont:

$$|\psi(4T)\rangle = e^{-iE_0 4T/\hbar} e^{i\gamma} |1\rangle$$ ← 3 points

what is γ ?

This Hamiltonian is exactly that for a spin 1/2 particle in a $\vec{B}$

$$\vec{B} \propto \cos\frac{\pi t}{2T}\,\hat{z} + \sin\frac{\pi t}{2T}\,\hat{x}$$

So, for $0 < t < 4T$, $\vec{B}$ sweeps once around the "equator" in the x-z plane, describing "cone" with solid angle 2π.

And, $\gamma = -\frac{1}{2}\Omega = -\pi$.

$\therefore\ e^{i\gamma} = e^{-i\pi} = -1$ ← 3 points

$$\therefore\ |\psi(4T)\rangle = -e^{-i4E_0T/\hbar}\,|1\rangle$$

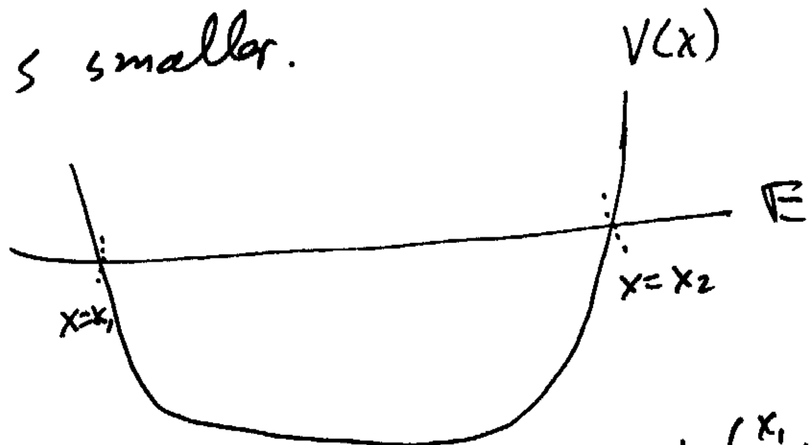
7. a) $\psi(x) = \frac{C_1}{\sqrt{p(x)}} e^{+\frac{i}{\hbar}\int^x p(x')dx'} + \frac{C_2}{\sqrt{p(x)}} e^{-\frac{i}{\hbar}\int^x p(x')dx'}$

where C_1 & C_2 are constants.

$\therefore |\psi|^2 \propto \frac{1}{p(x)} \propto \frac{1}{m v(x)}$

$\therefore$ probability to find the particle is greater where the velocity $v(x)$ is smaller.

b)



For $x \ll x_1$, solution must be $\frac{1}{\sqrt{\kappa(x)}} e^{-\frac{1}{\hbar}\int_x^{x_1} \kappa(x')dx'}$

$\Rightarrow$ for $x_1 \ll x \ll x_2$, solution must have form

$$\frac{2C}{\sqrt{p(x)}} \cos\left[\frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' - \frac{\pi}{4}\right]$$

7 b) cont.

For $x \gg x_2$, solution must be $\frac{1}{\sqrt{\kappa(x)}} e^{-\frac{1}{\hbar}\int_{x_2}^{x} \kappa(x')dx'}$

$\Rightarrow$ for $x_1 \ll x \ll x_2$, solution must have form

$$\frac{2c'}{\sqrt{p(x)}} \cos\left[\frac{1}{\hbar}\int_x^{x_2} p(x')dx' - \frac{\pi}{4}\right]$$

$$\therefore \quad \frac{1}{\hbar}\int_{x_1}^{x} p(x')\,dx' - \frac{\pi}{4} = -\frac{1}{\hbar}\int_x^{x_2} p(x')dx' + \frac{\pi}{4} + n\pi$$

where for n = even we choose $c' = c$
and " n = odd " " $c' = -c$

$$\therefore \quad \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')dx' = \left(n + \frac{1}{2}\right)\pi \quad \text{for } n = 0, 1, 2, \ldots$$

or, equivalently,

$$\int_{x_1}^{x_2} p(x')dx' = \left(n - \frac{1}{2}\right)\pi\hbar \quad \text{for } n = 1, 2, 3, \ldots$$

c) Probability ~ $e^{-\frac{2}{\hbar}\int_{\text{barrier}} \sqrt{2m(V_0 - E)}\,dx}$

where $\int_{\text{barrier}}$ means over the range in to labelled by $\longleftarrow w \longrightarrow$ in the figure.

7c) cont.

$$\text{Probability} \sim e^{-\frac{2}{\hbar} w \sqrt{2m(V_0-E)}}$$

i): Double m, Prob changes from 10^{-10}

to $10^{-10\sqrt{2}}$

ii) Double w, Prob changes from 10^{-10} to 10^{-20}

iii) " V_0-E, " " " 10^{-10} to $10^{-10\sqrt{2}}$

8. a) $\vec{B} = (0, 0, B_0)$ $\qquad$ $\vec{E} = (0, E_0, 0)$

$\vec{A} = (-B_0 y, 0, 0)$ $\qquad$ $\phi = -y E_0$

$$H = \frac{1}{2m}\left(\vec{P} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$$= \frac{1}{2m}\left(P_x + \frac{qB_0 y}{c}\right)^2 + \frac{P_y^2}{2m} - qyE_0$$

b) Since $[H, P_x] = 0$, we can look for simultaneous eigenstates of H & P_x, with P_x eigenvalue $\hbar k_x$ and wave function $e^{ik_x x} f(y)$. For such states,

$$H = \frac{P_y^2}{2m} + \frac{1}{2m}\left(\hbar k_x + \frac{qB_0 y}{c}\right)^2 - qyE_0$$

$$= \frac{P_y^2}{2m} + \frac{q^2 B_0^2}{2mc^2}\left(y + \frac{\hbar c k_x}{qB_0}\right)^2 - qyE_0$$

$$= \frac{P_y^2}{2m} + \frac{1}{2} m\omega_L^2 \left(y + \frac{\hbar c k_x}{qB_0}\right)^2 - qyE_0$$

$$= \frac{P_y^2}{2m} + \frac{1}{2} m\omega_L^2 \left(y + \frac{\hbar c k_x}{qB_0} - \frac{qE_0}{2\cdot\frac{1}{2}m\omega_L^2}\right)^2$$

$$- \frac{1}{2} m\omega_L^2 \frac{q^2 E_0^2}{m^2\omega_L^4} + 2\frac{\hbar c k_x}{qB_0}\frac{qE_0}{m\omega_L^2}\frac{1}{2}m\omega_L^2$$

8b) cont:

$$H = \frac{P_y^2}{2m} + \frac{1}{2} m \omega_L^2 \left(y + \frac{\hbar c k_x}{q B_0} - \frac{q E_0}{m \omega_L^2} \right)^2$$

$$- \frac{1}{2} \frac{q^2 E_0^2}{m \omega_L^2} + \frac{\hbar c k_x E_0}{B_0}$$

This is a S.H.O. centered at $y = y_0$

where $$y_0 = \frac{q E_0}{m \omega_L^2} - \frac{\hbar c k_x}{q B_0} = \frac{E_0 m c^2}{q B_0^2} - \frac{\hbar c k_x}{q B_0}$$

with energy eigenvalues

$$(n + \tfrac{1}{2}) \hbar \omega_L - \frac{1}{2} \frac{q^2 E_0^2}{m \omega_L^2} + \frac{\hbar c k_x E_0}{B_0}$$

$$= (n + \tfrac{1}{2}) \hbar \omega_L - \frac{1}{2} \frac{m E_0^2 c^2}{B_0^2} + \frac{\hbar c k_x E_0}{B_0}$$

$$= (n + \tfrac{1}{2}) \hbar \omega_L - \frac{1}{2} m v_D^2 + \frac{\hbar c k_x E_0}{B_0}$$

$$= (n + \tfrac{1}{2}) \hbar \omega_L + \frac{1}{2} m v_D^2 - m v_D^2 + \frac{\hbar c k_x E_0}{B_0}$$

$$= (n + \tfrac{1}{2}) \hbar \omega_L + \frac{1}{2} m v_D^2 - \frac{m c^2 E_0^2}{B_0^2} + \frac{\hbar c k_x E_0}{B_0}$$

$$= (n + \tfrac{1}{2}) \hbar \omega_L + \frac{1}{2} m v_D^2 - q E_0 a$$

for $$a = \frac{m c^2 E_0}{q B_0^2} - \frac{\hbar c k_x}{q B_0} = y_0 \,!$$

8b) cont.

So, the wave functions of the energy eigenstates are S.H.O. wave functions in y, centered at $y = a$.

c) No restriction on n. $(n = 0, 1, 2 \ldots)$

Restriction on a: $-L/2 < a < L/2$

so that wave functions are in the correct regime.

d) Still no restriction on n.

a now restricted to discrete values spaced by $\frac{2\pi}{W}$.

9 a) Low energy scattering is dominated by s-wave.

$$\therefore \quad f = \underbrace{P_0(\cos\theta)}_{1} \frac{e^{2i\delta_0} - 1}{2ik} = \frac{e^{2i\delta_0} - 1}{2ik}$$

For $k \to 0$, $\delta_0 \sim \tan\delta_0 \sim \sin\delta_0 \sim -ak$

$$\therefore \quad f \sim \frac{(1 + 2i(-ak) - 1)}{2ik} = -a$$ 4 points

$$\sigma = \frac{4\pi}{k^2} \sin^2\delta_0 = \frac{4\pi}{k^2}(-ak)^2 = 4\pi a^2$$ & 2 points

So, for $k \to 0$ $\boxed{\begin{aligned} f &= -a \\ \sigma &= 4\pi a^2 \end{aligned}}$

b) $$f_{Born} = -\frac{m}{2\pi\hbar^2} \int d^3r \, e^{-i\vec{q}\cdot\vec{r}} V(\vec{r})$$

$$= -\frac{m}{2\pi\hbar^2} \int d^3r \, e^{-i\vec{q}\cdot\vec{r}} \lambda \frac{\hbar^2}{2mb} \delta(r-b)$$

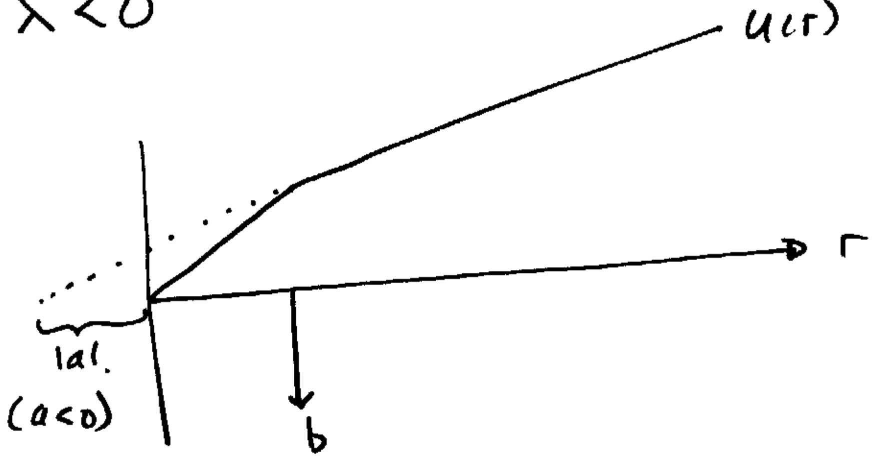
Since $|\vec{q}| = 2k\sin\frac{\theta}{2}$, $\lim_{k \to 0} e^{-i\vec{q}\cdot\vec{r}} = 1$

$\therefore$ In $k \to 0$ limit, $$f_{Born} = -\frac{m}{2\pi\hbar^2} \lambda \frac{\hbar^2}{2mb} 4\pi \int r^2 dr \, \delta(r-b)$$

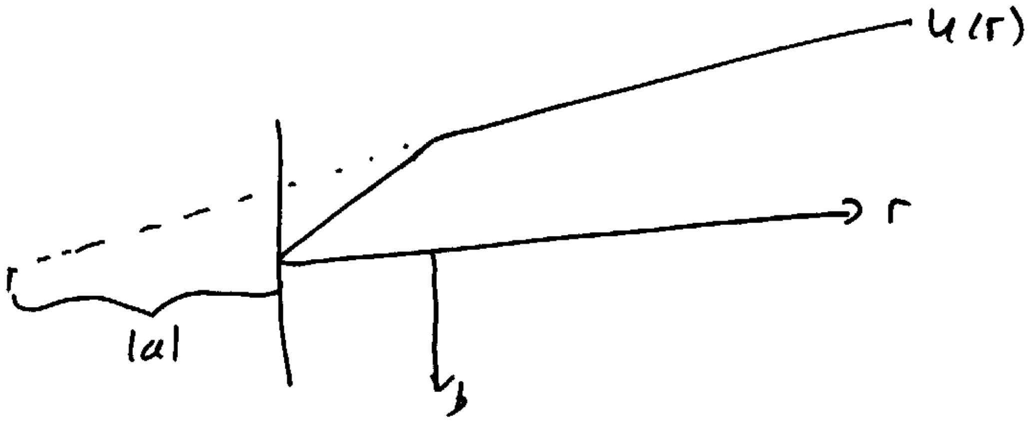
$$= -\frac{\lambda}{b} b^2 = -\lambda b$$

Since $f = -\lambda b$, and from (a) $f = -a$, $\Rightarrow \boxed{a = \lambda b}$ for small $|\lambda|$.

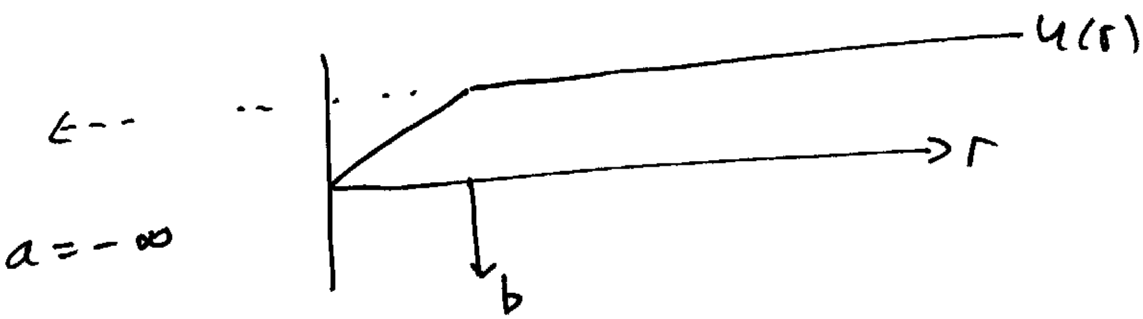
9 c) $\lambda < 0$



$\downarrow$ increase $|\lambda|$



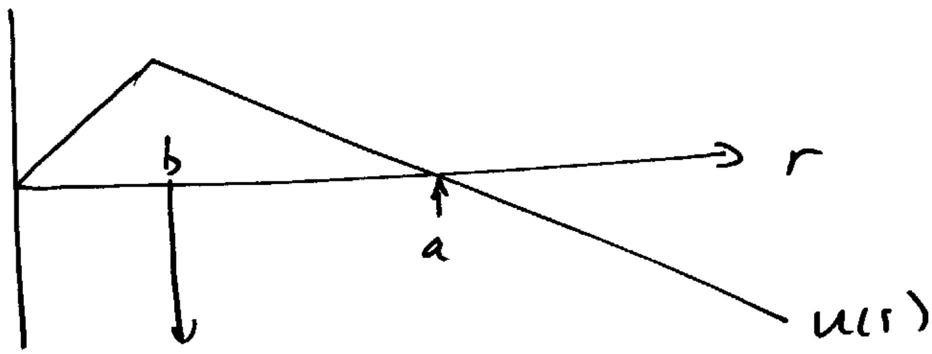
$\downarrow$ increase $|\lambda|$



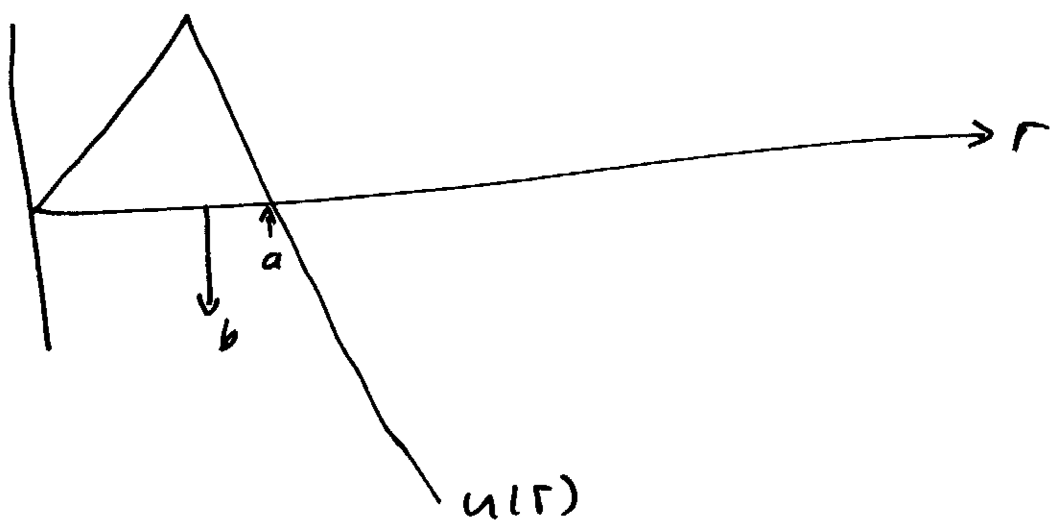
There is a special negative λ at which $a \to -\infty$.

9c) cont.

Keep increasing $|\lambda|$

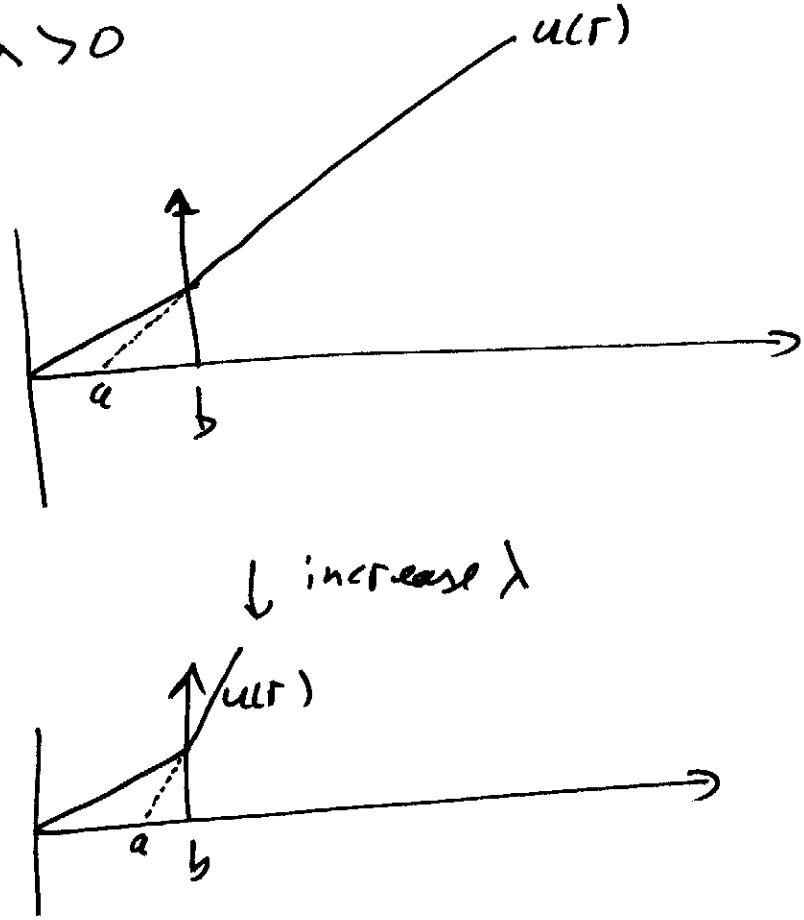


$\downarrow$ increase $|\lambda|$



As $|\lambda| \to \infty$, with $\lambda < 0$, a approaches b from above. There is only one "special value of $|\lambda|$".

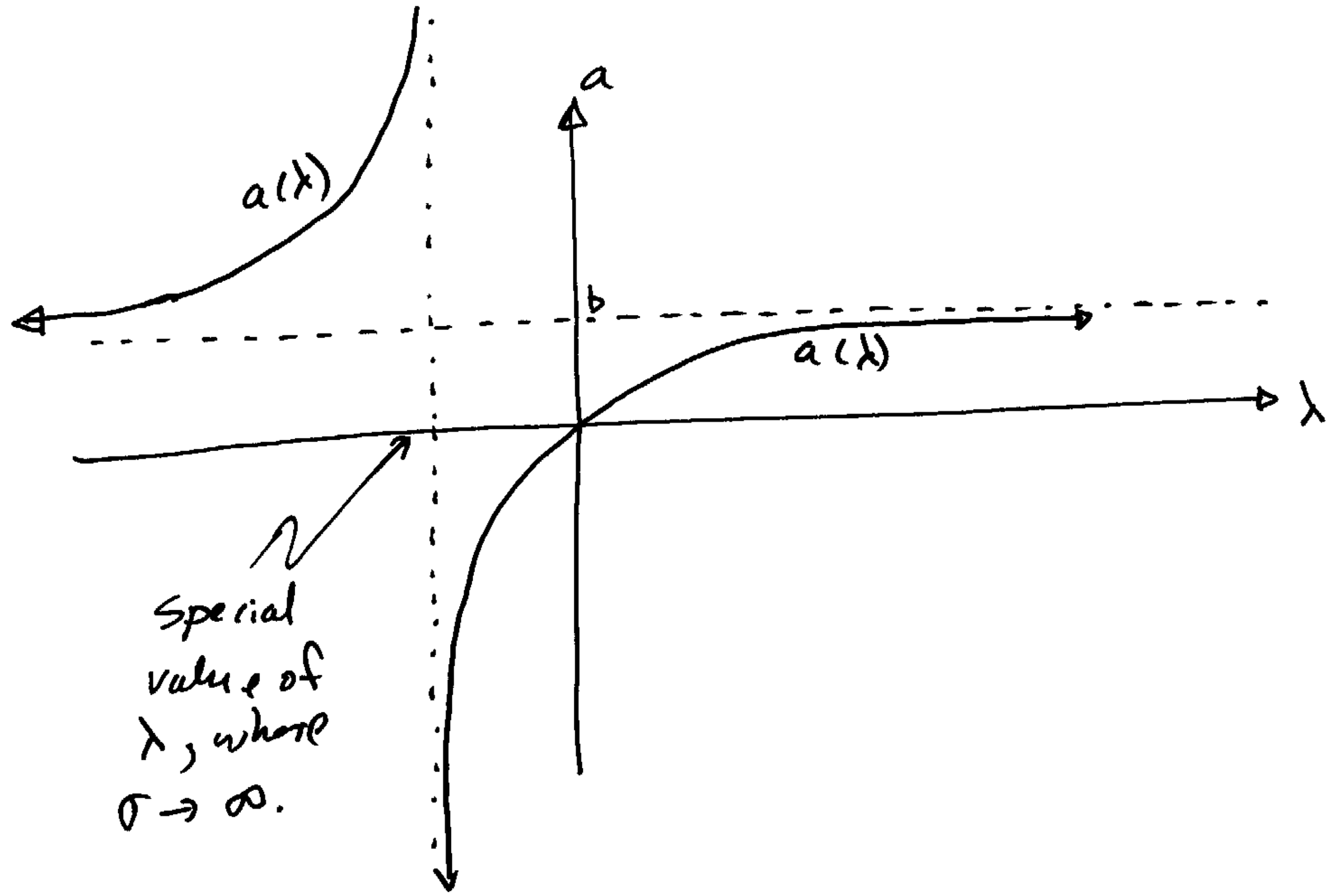
9 d) $\lambda > 0$



No special values of λ for $\lambda > 0$.

As $\lambda \to +\infty$, a ~~approa~~ approaches b from below.

9 e) From our graphical analysis ....



The exact calculation gives

$$a = b \frac{\lambda}{1+\lambda}$$

so the special value of λ is $\lambda = -1$.