

Quantum Physics III (8.06) Spring 2005
FINAL EXAMINATION
Tuesday May 17, 1:30 pm
You have 3 hours.

There are 12 problems, totalling 180 points. Do all problems.

Answer all problems in the blue books provided.

Write YOUR NAME on EACH blue book you use.

Budget your time wisely, using the point values as a guide. Note also that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

- Conservation of Probability

$$\frac{\partial}{\partial t} \rho(\vec{r}, t) + \vec{\nabla} \cdot \vec{J}(\vec{r}, t) = 0$$

where

$$\rho(\vec{r}, t) = |\psi(\vec{r}, t)|^2 ; \quad \vec{J}(\vec{r}, t) = \frac{\hbar}{2im} [\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*]$$

- Operators for Spin-1/2 particle

$$\hat{S}_i = \frac{\hbar}{2} \sigma_i$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar \omega \left(\hat{N} + \frac{1}{2} \right)$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} + i\hat{p}) , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} - i\hat{p}) ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left(-\frac{m\omega}{2\hbar} x^2 \right) .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

- Spherical Coordinates

$$x = r \sin \theta \cos \phi ; \quad y = r \sin \theta \sin \phi ; \quad z = r \cos \theta$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

- Angular Momentum

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k ; \quad [\hat{L}^2, \hat{L}_i] = 0$$

$$\hat{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle ; \quad \hat{L}_z |\ell, m\rangle = \hbar m |\ell, m\rangle$$

$$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$$

$$[\hat{L}_{\pm}, \hat{L}^2] = 0 ; \quad [\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$$

$$\hat{L}^2 = \hat{L}_+ \hat{L}_- + \hat{L}_z^2 - \hbar \hat{L}_z$$

$$\hat{L}_{\pm} |\ell, m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m \pm 1)} |\ell, m \pm 1\rangle$$

- Angular momentum operators in spherical coordinates

$$\hat{L}^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} ; \quad \hat{L}_{\pm} = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right)$$

- Spherical Harmonics

$$Y_{\ell, m}(\theta, \phi) \equiv \langle \theta, \phi | \ell, m \rangle$$

$$Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} ; \quad Y_{1,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta \exp(\pm i\phi) ; \quad Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{2,\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta \exp(\pm 2i\phi) ; \quad Y_{2,\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta \exp(\pm i\phi) ;$$

$$Y_{2,0}(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

- Hydrogen atom

Energy eigenvalues are $E_n = -E_I/n^2$ where n , the principal quantum number, is a positive integer, and $E_I = me^4/2\hbar^2 \sim 13.6$ eV.

Bohr radius is $a_0 = \hbar^2/me^2$

Wave functions for states which are eigenstates of H , L^2 and L_z have the form $\psi_{n,\ell,m}(\vec{r}) = \frac{1}{r} u_{n,\ell}(r) Y_{\ell,m}(\theta, \phi)$, where

$$\begin{aligned} u_{n=1,\ell=0}(r) &= \frac{2r}{a_0^{3/2}} \exp(-r/a_0) \\ u_{n=2,\ell=0}(r) &= \frac{2r}{(2a_0)^{3/2}} \left(1 - \frac{r}{2a_0}\right) \exp(-r/2a_0) \\ u_{n=2,\ell=1}(r) &= \frac{1}{\sqrt{3}} \frac{1}{(2a_0)^{3/2}} \frac{r^2}{a_0} \exp(-r/2a_0) \end{aligned}$$

- Some useful constants:

$$\hbar c = 197 \times 10^{-7} \text{ eV cm}$$

The mass of the electron is $m_e = 0.511 \text{ MeV}/c^2$. If B is 1 gauss, then the force eB is 300 eV/cm.

- Particle in an Electric and/or Magnetic Field:

The Hamiltonian for a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and an electric field $\vec{E} = -\vec{\nabla}\phi$ is:

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\phi \quad (1)$$

- Gauge invariance:

If $\psi(\vec{x}, t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x}, t) = \exp\left(\frac{iq}{\hbar c} f(\vec{x}, t)\right) \psi(\vec{x}, t)$$

solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla}f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

Suppose that

$$H = H^0 + H'$$

where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0 :

$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle .$$

Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^0 + H'_{nn} + \sum_{m \neq n} \frac{|H'_{nm}|^2}{E_n^0 - E_m^0} + \dots$$

$$|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m \neq n} \frac{H'_{mn}}{E_n^0 - E_m^0} |\psi_m^0\rangle + \dots$$

where $H'_{nm} \equiv \langle \psi_n^0 | H' | \psi_m^0 \rangle$.

- Connection Formulae for WKB Wave Functions:

At a turning point at $x = a$ at which the classically forbidden region is at $x > a$:

$$\begin{aligned} \frac{2}{\sqrt{p(x)}} \cos \left[\frac{1}{\hbar} \int_x^a p(x') dx' - \frac{\pi}{4} \right] &\leftarrow \frac{1}{\sqrt{\kappa(x)}} \exp \left[-\frac{1}{\hbar} \int_a^x \kappa(x') dx' \right] \\ \frac{1}{\sqrt{p(x)}} \cos \left[\frac{1}{\hbar} \int_x^a p(x') dx' + \frac{\pi}{4} \right] &\rightarrow \frac{1}{\sqrt{\kappa(x)}} \exp \left[+\frac{1}{\hbar} \int_a^x \kappa(x') dx' \right] \end{aligned}$$

At a turning point at $x = b$ at which the classically forbidden region is at $x < b$:

$$\begin{aligned} \frac{1}{\sqrt{\kappa(x)}} \exp \left[-\frac{1}{\hbar} \int_x^b \kappa(x') dx' \right] &\rightarrow \frac{2}{\sqrt{p(x)}} \cos \left[\frac{1}{\hbar} \int_b^x p(x') dx' - \frac{\pi}{4} \right] \\ \frac{1}{\sqrt{\kappa(x)}} \exp \left[+\frac{1}{\hbar} \int_x^b \kappa(x') dx' \right] &\leftarrow \frac{1}{\sqrt{p(x)}} \cos \left[\frac{1}{\hbar} \int_b^x p(x') dx' + \frac{\pi}{4} \right] \end{aligned}$$

- Born Approximation to Scattering Amplitude:

$$f(\theta, \phi) = f(\vec{q}) = -\frac{m}{2\pi\hbar^2} \int d^3r \exp(-i\vec{q} \cdot \vec{r}) V(\vec{r})$$

where $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transfer.

- Partial Wave Analysis of Scattering from a spherically symmetric potential:

$$f(\theta) = \sum_l (2\ell + 1) P_\ell(\cos \theta) \frac{\exp(2i\delta_\ell) - 1}{2ik}$$

$$\sigma = \frac{4\pi}{k^2} \sum_\ell (2\ell + 1) \sin^2 \delta_\ell$$

- Legendre Polynomials

$$\begin{aligned} P_0(z) &= 1 \\ P_1(z) &= z \\ P_2(z) &= \frac{3z^2 - 1}{2} \\ P_3(z) &= \frac{5z^3 - 3z}{2} \end{aligned}$$

- Time Dependent Perturbation Theory:

Consider a system with the Hamiltonian

$$H = H_0 + V(\vec{r}) \cos(\omega t) \exp(-t^2/T^2)$$

and denote the matrix element of V between eigenstates of H_0 named $|a\rangle$ and $|b\rangle$ by V_{ab} . Then, in the large T limit, if the system is initially in state $|a\rangle$, the probability that it is in the state $|b\rangle$ for $t \rightarrow +\infty$ is:

$$P_{a \rightarrow b} = \frac{\pi}{2\hbar^2} |V_{ab}|^2 \delta(\omega - \omega_{ab}) \sqrt{\pi} T$$

where $\hbar\omega_{ab} = E_b - E_a$.

1. True or False? (16 points)

Is each of the following statements true or false? If true, simply write the word “True”. If false, you must *briefly* explain why or in what way the statement it is false.

- (a) The relativistic correction, spin-orbit correction, and Lamb shift are all effects on the spectrum of hydrogen that are smaller than the spacing between Bohr levels by a factor of v^2/c^2 , v being the velocity of the electron in a hydrogen atom.
- (b) The hydrogen atom can never have an electric dipole moment, since none of its energy eigenstates have electric dipole moments.
- (c) The energy eigenstates of a hydrogen atom in a very strong magnetic field are eigenstates of J^2 and J_z .
- (d) The energy eigenstates of a hydrogen atom in a very weak magnetic field are eigenstates of J^2 and J_z .
- (e) A hydrogen atom excited into its $2s$ -state decays back to the ground state by the spontaneous emission of a photon much more quickly than a hydrogen atom excited into one of its $2p$ -states does.
- (f) In analyzing the “Landau problem” — namely the motion of an electron in a two-dimensional (x, y) -plane that feels a magnetic field perpendicular to the plane — it is always possible to choose a gauge such that the Hamiltonian commutes with both p_x and p_y .
- (g) The degeneracy of a Landau level increases with increasing magnetic field, and so does the separation in energy between Landau levels.
- (h) If the two-dimensional plane is finite in extent, say L long by W wide, and if a small but nonzero electric field is applied in a direction parallel to the plane, then the Landau level degeneracy is completely broken: no energy level is degenerate.

2. Mini-Essay (20 points)

Write a *short* description — say one paragraph of text, possibly with sketched figures in addition — of **ONE** of the following topics. Using equations is neither required nor forbidden. However, you should focus on a clear description in words and perhaps pictures. If you do draw figures, make sure to label them clearly.

- (a) The Integer Quantum Hall Effect
- (b) The MSW Mechanism and the Solar Neutrino Problem
- (c) The Born-Oppenheimer Approximation
- (d) The Aharonov-Bohm Effect

To repeat, you need only write about **ONE** of the above topics.

3. Dimensional Analysis of a Degenerate Fermi Gas (10 points)

A gas of fermions at zero temperature has a pressure P proportional to the number density (n) raised to some power (γ):

$$P = Kn^\gamma .$$

In general, the constant K may depend on three constants: $\hbar$, c , and the fermion mass m .

- (a) (5 points) On which of the above three constants does K depend for a non-relativistic Fermi gas? Use dimensional analysis to find the numerical value of γ , and the dependence of K on the constants, for a non-relativistic Fermi gas.
- (b) (5 points) On which of the above three constants does K depend for an ultra-relativistic Fermi gas? Use dimensional analysis to find the numerical value of γ , and the dependence of K on the constants, for an ultra-relativistic Fermi gas.

4. Stark Effect in Hydrogen (10 points)

Consider the first excited state of hydrogen. (The $n = 2$ level.) In this problem, you may neglect all fine and hyperfine structure, and may neglect spin. You need not show any calculations.

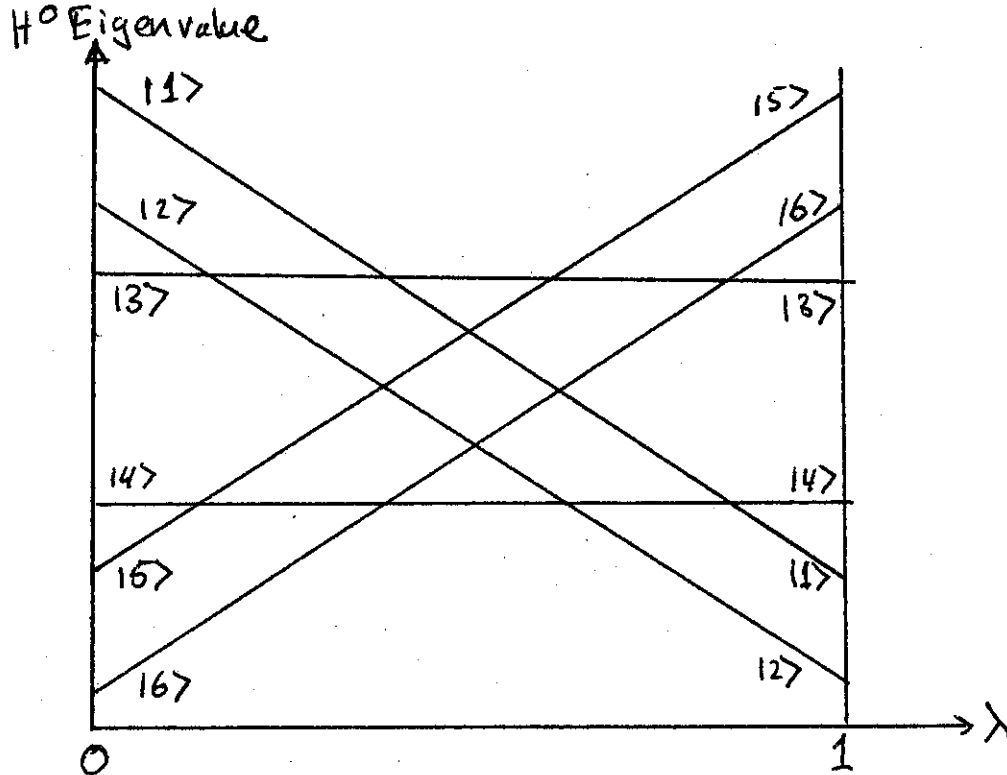
- (a) (2 points) List the states in the $n = 2$ level, using the standard $|n\ell m\rangle$ notation.
- (b) (8 points) In the presence of a constant applied electric field in the z -direction, the degeneracy among these states is broken. What are the $n = 2$ eigenstates of the Hamiltonian in the presence of a weak applied electric field? Which of these states, if any, have the same energy as they would in the absence of an electric field?

5. A Random Perturbation of a Complex System (5 points)

The Hamiltonian for a certain six-state system depends on a single parameter λ which takes on values between 0 and 1. This Hamiltonian can be written as

$$H = H^0 + H'.$$

The eigenvalues of H^0 are shown as a function of λ in the following sketch, with each eigenvalue labelled by the name of the corresponding eigenstate:



The 36 matrix elements of H' are all nonzero, are all very small, and have been chosen at random, subject only to the condition that H' is Hermitian.

As time progresses from $-\infty$ to ∞ , λ changes very slowly from 0 to 1.

Suppose that the system is in one of the six energy eigenstates at $t = -\infty$. For each of the six initial possibilities, what is the state of the system at $t = +\infty$?

6. Adiabatic Evolution in a Time-Dependent Two-State System (13 points)

Consider a two-state system with basis states $|1\rangle$ and $|2\rangle$, and with a Hamiltonian given in this basis by

$$H = A\sigma_3 + B\sigma_1$$

- (a) (6 points) Suppose B is constant and $A(t)$ is a slowly varying function of time. Suppose the system is in the state $|1\rangle$ in the distant past. Explain how to choose a function $A(t)$ such that the system transforms to the state $|2\rangle$ in the distant future. Give an example of such a function $A(t)$. What combinations of parameters (including parameters that you use in the specification of your $A(t)$) must be small in order that your transformation works?
- (b) (7 points) Now suppose both A and B vary with time, with $A(t) = K \cos ft$ and $B(t) = K \sin ft$ where K and f are constants. Assume that f is very small, and the adiabatic approximation can therefore be applied. Suppose the system is in the state $|1\rangle$ at $t = 0$. What is the phase of the state at time $T = 2\pi/f$ compared to the phase at $t = 0$? Be sure to include both the dynamical phase and the Berry phase.

7. WKB on a Ring (10 points)

A particle of mass m moves on a ring of circumference L . It is subject to a potential $V(s)$, for $0 < s < L$. Here, s is the coordinate specifying position on the ring. Of course, $V(0) = V(L)$. Assume that the particle has a large enough energy E that $E > V(s)$ everywhere on the ring.

- (a) (5 points) What is the quantization condition on the energy eigenvalue E , in the WKB approximation?
- (b) (5 points) Within the WKB approximation, how many degenerate states are there for each energy eigenvalue? What is (are) the wave function(s) for the state(s) with eigenvalue E ? [You need not normalize the wave functions.]

8. Low Energy Scattering, in Pictures (12 points)

Consider s -wave scattering off an attractive central potential in three dimensions, which we take to be simply $V = -V_0$ for $r \leq b$ and $V = 0$ for $r > b$.

- (a) (7 points) There are certain special values of V_0 for which the low energy scattering cross section for scattering from this potential diverges. Make a clear sketch showing the radial wave function $u(r)$ which describes scattering in the zero-energy limit for one of these special values of V_0 . As V_0 is changed from just above this special value to just below, what qualitative change occurs in the spectrum of the Hamiltonian?
- (b) (5 points) There are certain other special values of V_0 which are nonzero and for which the low energy scattering cross section for this potential nevertheless vanishes. Make a clear sketch showing the radial wave function $u(r)$ which describes scattering in the zero-energy limit for one of these special values of V_0 .

9. A Scattering Cross Section (12 points)

An experimentalist studies scattering from a central potential $V(r)$ in three dimensions. After lots of work, she measures a cross section as a function of scattering angle θ and momentum k , where $k^2 = 2mE/\hbar^2$, and finds:

$$\frac{d\sigma}{d\Omega} = A(k) + B(k) \cos \theta + C(k) \cos^2 \theta$$

- (a) (3 points) What partial waves (i.e. waves with what values of orbital angular momentum quantum number ℓ) are scattering?
- (b) (5 points) Write $A(k)$, $B(k)$ and $C(k)$ in terms of phase shifts δ_ℓ .
- (c) (4 points) What is the total cross section?

10. **A Variational Problem (22 points)**

A particle of mass m moves in the one dimensional potential $V(x) = \lambda x^p$, where p is a positive even integer.

- (a) (18 points) Use the trial wave functions

$$\psi(x) = \exp(-x^2/a^2) ,$$

where a is a parameter with the dimensions of length, to obtain an upper bound on the ground state energy.

You should obtain an answer written in terms of dimensionful parameters of the problem and constants of nature, some of which will be raised to powers that depend on p , and a dimensionless coefficient. You may leave the dimensionless coefficient written in terms of dimensionless integrals. In other words, do *not* attempt to evaluate any *dimensionless* integrals that arise in your calculation.

- (b) (4 points) Give an example of a set of trial wave functions, specified by a single parameter, that could be used to obtain an upper bound on the first excited state energy. What condition must these trial wave functions satisfy, and why?

11. **Perturbations of a Two-Dimensional Harmonic Oscillator (30 points)**

Consider a quantum system described by the Hamiltonian

$$H = H^0 + H'$$

where H^0 is the two-dimensional harmonic oscillator Hamiltonian

$$H^0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) .$$

In parts (a) and (b) of the problem, the perturbing Hamiltonian H' is given by

$$H' = Kp_x p_y ,$$

with K a constant.

- (a) (15 points) Evaluate the ground state energy to second order in K .
(b) (8 points) Evaluate the energy of the state(s) whose unperturbed energy (ie energy if K were zero) is $2\hbar\omega$ to first order in K .
(c) (7 points) Now set $K = 0$ and consider a new, time-dependent, perturbing Hamiltonian

$$H' = Kp_y^3 \cos \Omega t \exp(-t^2/T^2)$$

where $T \gg 1/\Omega$ and $T \gg 1/\omega$. Suppose that initially (i.e. at $t \rightarrow -\infty$) the system is in its ground state. For what value(s) of the perturbing frequency Ω will there be a nonzero probability that the system makes a transition out of its ground state? For each such frequency, to which excited state(s) are transitions possible?

12. **WKB Approximation for Scattering from a Central Potential (20 points)**

Suppose $V(r)$ is a smooth, deep, central potential in three dimensions which is everywhere attractive. Let $U(r) = -2mV(r)/\hbar^2$. Assume $U(r) \rightarrow 0$ quickly as $r \rightarrow \infty$. The goal of this problem is to use the WKB approximation to study the positive energy solutions with $\ell = 0$ in this potential. Define $k = \sqrt{2mE/\hbar^2}$ as usual.

- (a) (3 points) Write the Schrödinger equation for the radial wave function, $u(r)$, and state any boundary conditions on u . Remember, $\ell = 0$ and $E > 0$.
- (b) (9 points) Use the WKB approximation to find an approximate expression for the scattering phase shift $\delta_0(k)$.
- (c) (4 points) Show that your result from part (c) *does not* have the expected behavior, $\delta_0(k) \sim -ka$, as $k \rightarrow 0$. What has gone wrong?
- (d) (4 points) Find the leading behavior of $\delta_0(k)$ as $k \rightarrow \infty$. Explain why you do or do not expect the result you have found to be a good approximation.

OPTIONAL, FOR EXTRA CREDIT: Compare your $k \rightarrow \infty$ result to that obtained from the Born approximation to the spherical average of the scattering amplitude. [Note: maximum extra credit that will be given is 10 points, even though doing this may take you more than 10 points worth of time. It *is* optional.]