8.06 SPRING 2005

FINAL EXAM SOLUTION SET

1 a) FALSE. Lamb shift is $\mathcal{O}(v^3/c^3)$ smaller than spacing between Bohr levels

b) FALSE. $|210\rangle \pm |200\rangle$ ~~is~~ are energy eigenstates, and have electric dipole moments

c) FALSE. They are eigenstates of L_z & S_z

d) TRUE.

e) FALSE. Lifetime of 2s state <u>much</u> longer than that of 2p. $2p \to 1s$ goes by electric dipole emission. $2s \to 1s$ does not

f) FALSE. Since $\vec{B} = (0,0,B)$, $\vec{A}$ must either have A_x a function of y, or A_y a function of x, or both. That means x or y or both occurs in H. That means $[H, P_x] \neq 0$ or $[H, P_y] \neq 0$ or both.

g) TRUE

h) TRUE

2. No single solution set provided. Many different styles of mini-essay ~~but~~ got full credit, or close to full credit. This problem was graded by Prof. Rajagopal

3. $[P] = \left[\frac{F}{A}\right] = \frac{g\,cm}{s^2 cm^2} = \frac{g}{cm\,s^2}$

$[n] = \frac{1}{cm^3}$

$[\hbar] = \frac{g\,cm^2}{s}$

$[c] = cm/s$

$[m] = g$

a) K can depend on $\hbar$, m. Not on c.

$$\therefore P = const \times \underset{\substack{\uparrow \\ \text{to get} \\ 1/s^2}}{\hbar^2} \times \underset{\substack{\uparrow \\ \text{to then} \\ \text{get } g}}{\frac{1}{m}} \times \underset{\substack{\uparrow \\ \text{to get } 1/cm}}{n^{5/3}}$$

$\therefore K = const \times \frac{\hbar^2}{m}$ & $\gamma = 5/3$

b) K can depend on $\hbar$, c. Not on m

$$\therefore P = const \times \underset{\substack{\uparrow \\ \text{to get} \\ g}}{\hbar} \times \underset{\substack{\uparrow \\ \text{to get} \\ 1/s^2}}{c} \times n^{4/3}$$

$\therefore K = const \times \hbar c$ & $\gamma = 5/3$

4a) $|200\rangle$

$|211\rangle$

$|210\rangle$

$|21\,-1\rangle$

b) $H' \propto z$

need $\langle 2\,l m | z | 2 l' m' \rangle$

z: Parity odd. $\therefore$ need $|l - l'| = 1$

z: $e^{im\phi}$ for $m=0$; $\therefore$ need $m = m'$

$\therefore$ only nonzero matrix element is

$\langle 210 | z | 200 \rangle$

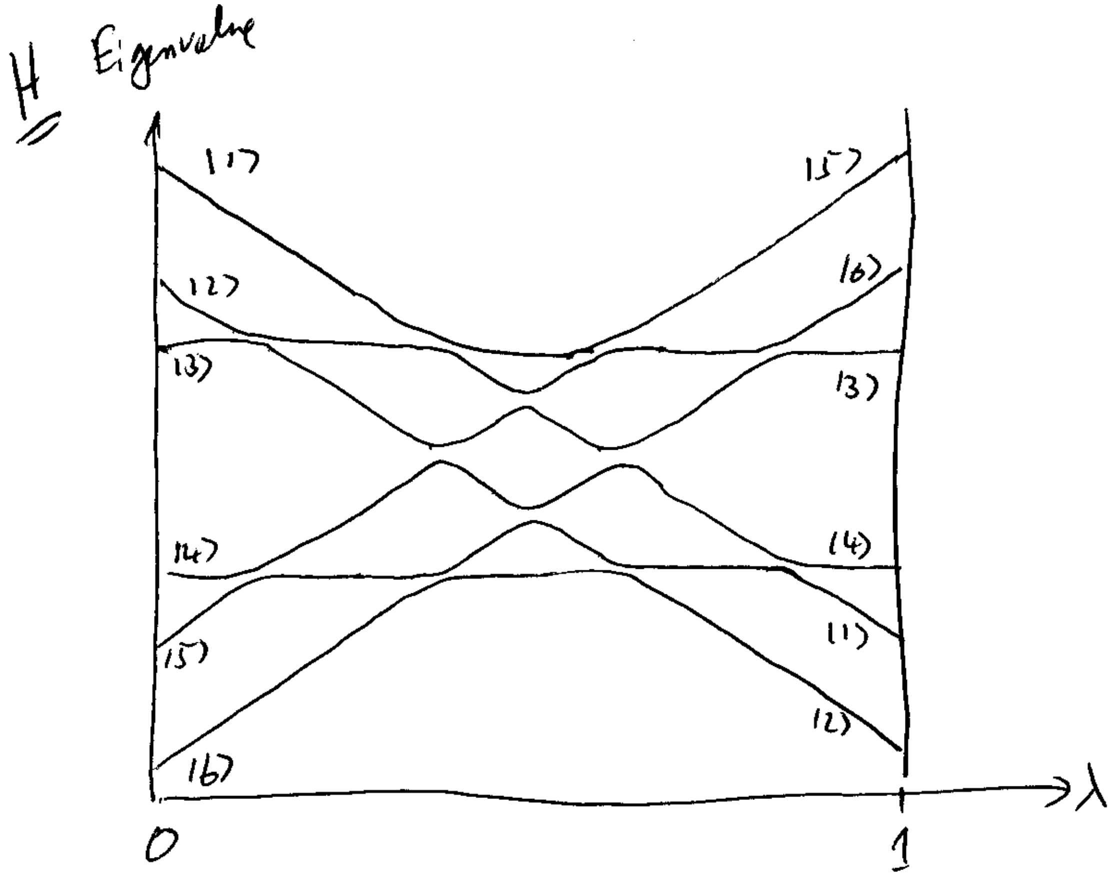
$\therefore$ Eigenstates

$$\left.\begin{array}{l} |211\rangle \\ |21\,-1\rangle \end{array}\right\} \text{energy not shifted by electric field}$$

$|210\rangle + |200\rangle$

$|210\rangle - |200\rangle$

5. The effect of the small off diagonal matrix elements in H' is to change the level diagram to:



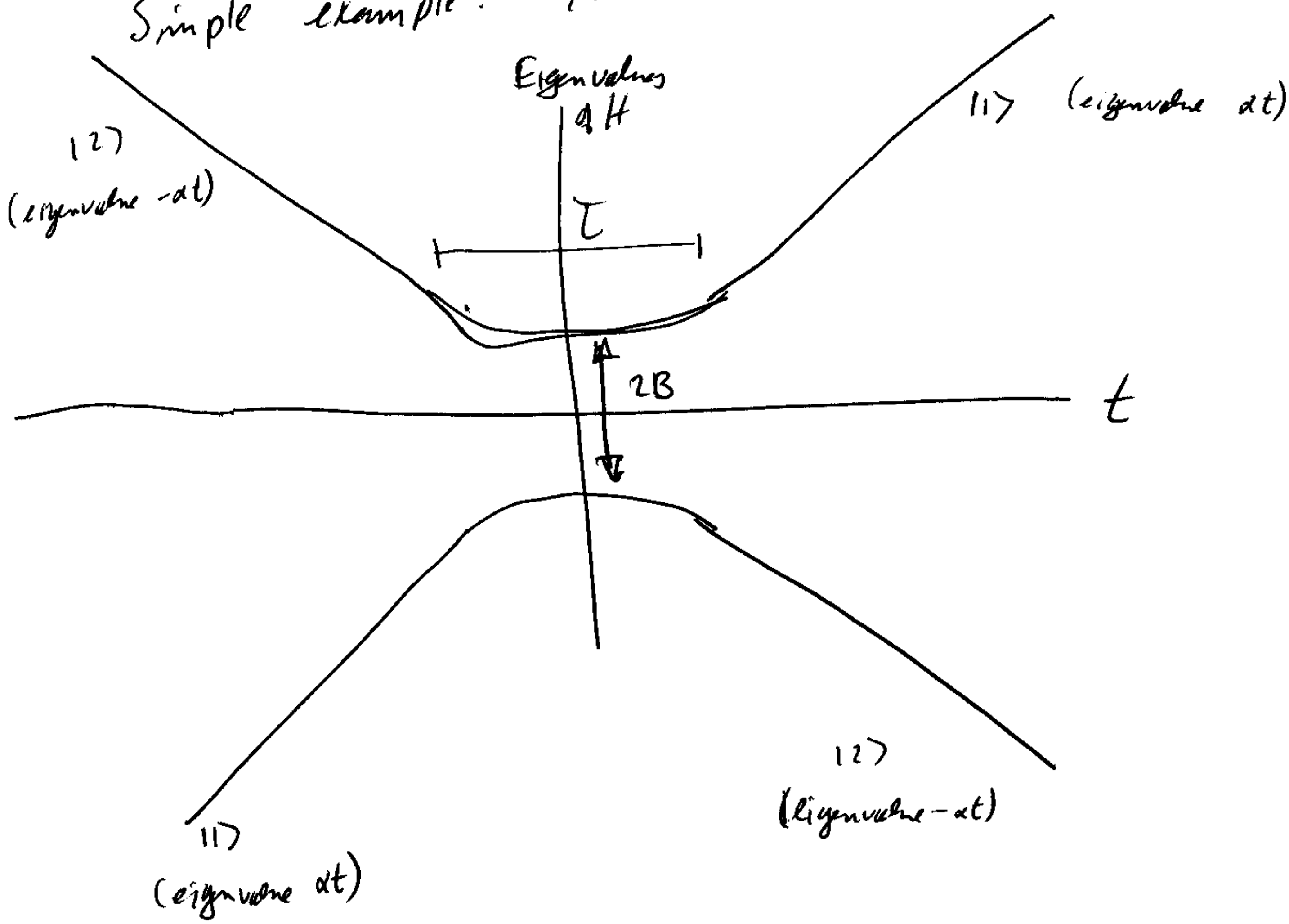
$$\therefore \quad \left.\begin{array}{l} |1\rangle \to |5\rangle \\ |2\rangle \to |6\rangle \\ |3\rangle \to |3\rangle \\ |4\rangle \to |4\rangle \\ |5\rangle \to |1\rangle \\ |6\rangle \to |2\rangle \end{array}\right\}$$

Full credit for this without the above figure

6 a)

$|A|$ has to be $\gg |B|$ in both distant past and distant future. And, sign of A has to be opposite in distant past and distant future.

Simple example: $A(t) = \alpha t$



Condition: $\tau \gg \frac{\hbar}{2B}$, where $\tau \simeq \frac{2B}{\alpha}$

$\therefore \frac{2B}{\alpha} \gg \frac{\hbar}{2B} \quad \therefore \alpha \ll \frac{(2B)^2}{\hbar}$

(Full credit if 2's dropped.)

6 b) At $t=0$, $H = K\sigma_3$.

$|1\rangle$ has energy $E = +K$

Eigenvalues same at all times. $\therefore E = +K$ at all ~~tq~~ times.

$\therefore$ Dynamical Phase $= e^{-\frac{i}{\hbar}\int_0^{2\pi/f} K\,dt}$

$= e^{-2\pi i K/\hbar f}$

Berry phase $= e^{-i\Omega/2}$

$\Omega =$ solid angle enclosed $= 2\pi$

~~$\therefore$ total phase $= e^{i\pi + \frac{i}{\hbar}}$~~

$\therefore$ total phase $= e^{-i\pi(2K/f\hbar + 1)}$

$= -e^{-2\pi i K/\hbar f}$

7a)

$$\frac{1}{\hbar}\int_0^L p\,ds = 2n\pi \quad , n=0,1,2,3,\ldots$$

$$\frac{1}{\hbar}\int_0^L \underbrace{\sqrt{2m(E-V(s))}}_{\equiv p(s)}\,ds = 2n\pi$$

b) Two degenerate states

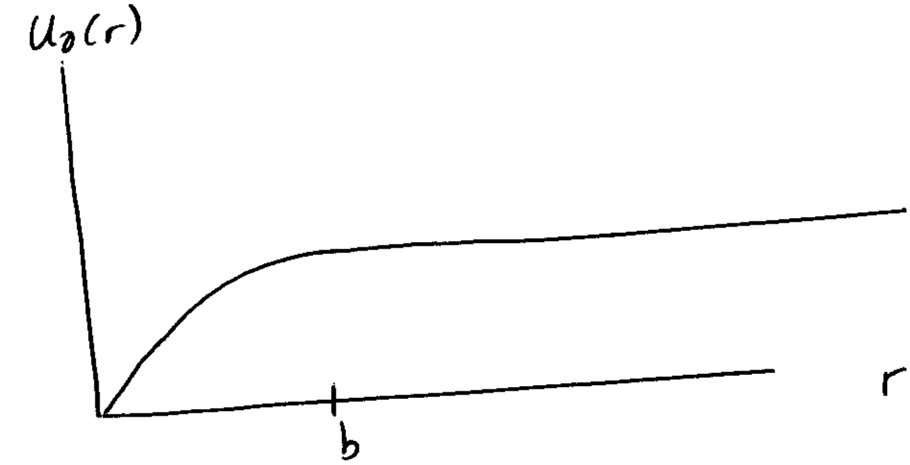
$$\psi_+ = \frac{C_+}{\sqrt{p(s)}}\, e^{+\frac{i}{\hbar}\int_0^s \sqrt{2m(E-V(s'))}\,ds'}$$

$$\psi_- = \frac{C_-}{\sqrt{p(s)}}\, e^{-\frac{i}{\hbar}\int_0^s \sqrt{2m(E-V(s'))}\,ds'}$$

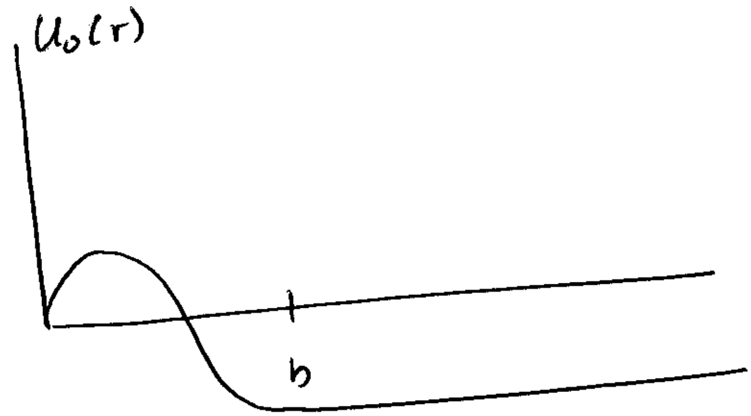
C_+ & C_- are undetermined constants.

(actually, ~~detd~~ can be determined by normalization but you were not asked for that.)

8 a)



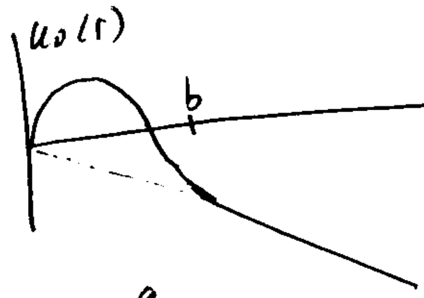
or



As the potential goes from just above to just below this special value $(-V_0)$, ~~the~~ the spectrum of the Hamiltonian gains a bound state.

[Full ~~and~~ credit for saying "As $\underline{V_0}$ .... the ~~potential~~ Hamiltonian loses a bound state." as long as it is clear.]

b)



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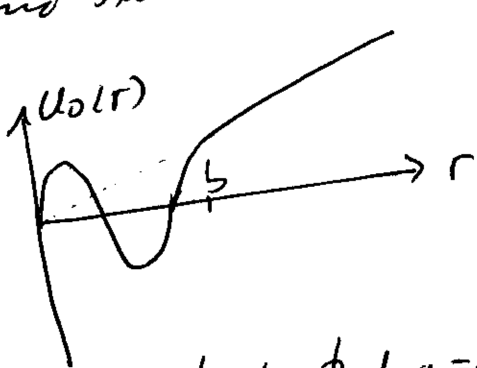


Diagram must show clearly that $a=0$.

9 a) $l=0$ & $l=1$

b) $$\frac{d\sigma}{d\Omega} = \left| \frac{e^{2i\delta_0}-1}{2ik} + 3\cos\theta\,\frac{e^{2i\delta_1}-1}{2ik} \right|^2$$

$$\therefore\ A(k) = \left| \frac{e^{2i\delta_0}-1}{2ik} \right|^2$$

$$= \left| \frac{e^{i\delta_0}\sin\delta_0}{k} \right|^2$$

$$= \frac{\sin^2\delta_0}{k^2}$$

$$C(k) = \frac{9\sin^2\delta_1}{k^2}$$

$$B(k) = 3\left(\frac{e^{2i\delta_0}-1}{2ik}\cdot\frac{e^{-2i\delta_1}-1}{-2ik} + \frac{e^{-2i\delta_0}-1}{-2ik}\cdot\frac{e^{2i\delta_0}-1}{2ik} \right)$$

$$= \frac{3}{k^2}\sin\delta_0\sin\delta_1\left(e^{i(\delta_0-\delta_1)} + e^{-i(\delta_0-\delta_1)}\right)$$

$$= \frac{6}{k^2}\sin\delta_0\sin\delta_1\cos(\delta_0-\delta_1)$$

c) $$\sigma = \frac{4\pi}{k^2}\left(\sin^2\delta_0 + 3\sin^2\delta_1\right) = \frac{4\pi}{k^2}\left(k^2A(k) + \frac{1}{9}k^2C(k)\right)$$

$$= 4\pi\left(A(k) + \frac{1}{3}C(k)\right)$$ [Can also get this by $\int d\Omega \frac{d\sigma}{d\Omega}$]

10. a)

$$E[a] \equiv \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} dx\, e^{-2x^2/a^2} = a\overbrace{\int_{-\infty}^{\infty} dy\, e^{-2y^2}}^{C_1}$$

$$= a \cdot C_1$$

Throughout, $C_1, C_2, \ldots$ will be dimensionless integrals, or combinations thereof.

$$\langle\psi|\lambda x^p|\psi\rangle = \lambda\int_{-\infty}^{\infty} dx\, x^p e^{-2x^2/a^2} = \lambda a^{p+1}\underbrace{\int_{-\infty}^{\infty} dy\, y^p e^{-2y^2}}_{C_2}$$

$$= \lambda a^{p+1} C_2$$

$$\langle\psi|-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}|\psi\rangle = \int_{-\infty}^{\infty} dx\, \frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|^2$$

$$= \int_{-\infty}^{\infty} dx\, \frac{\hbar^2}{2m}\left(\frac{-2x}{a^2}\right)^2 e^{-2x^2/a^2}$$

$$= \frac{2\hbar^2}{a^4 m} a^3 \underbrace{\int_{-\infty}^{\infty} dy\, y^2 e^{-2y^2}}_{C_3}$$

$$= \frac{2\hbar^2}{ma} C_3$$

$$\therefore\ E[a] = \frac{\frac{2\hbar^2}{ma}C_3 + \lambda a^{p+1} C_2}{a\, C_1} = \frac{2\hbar^2 C_3}{ma^2 C_1} + \frac{\lambda a^p C_2}{C_1}$$

$$\frac{dE\{a\}}{da} = 0 = -\frac{4\hbar^2 C_3}{m a^3 C_1} + \frac{\lambda p a^{p-1} C_2}{C_1}$$

$$\therefore \quad a^{p+2} = \frac{4\hbar^2 C_3}{\lambda m p C_2} = C_4 \frac{\hbar^2}{\lambda m}$$

with $C_4 \equiv \dfrac{4C_3}{pC_2}$

and $a = \left(C_4 \dfrac{\hbar^2}{\lambda m}\right)^{\frac{1}{p+2}}$ is best a.

$$\therefore \quad E_{gs} \leq \frac{2\hbar^2}{m}\frac{C_3}{C_1}\left(C_4\frac{\hbar^2}{\lambda m}\right)^{\frac{-2}{p+2}} + \lambda\frac{C_2}{C_1}\left(C_4\frac{\hbar^2}{\lambda m}\right)^{\frac{p}{p+2}}$$

$$= C_5 \frac{\hbar^2}{m}\left(\frac{\hbar^2}{\lambda m}\right)^{-2/p+2} + C_6\lambda\left(\frac{\hbar^2}{\lambda m}\right)^{\frac{p}{p+2}}$$

$$= C_7 \left(\frac{\hbar^2}{m}\right)^{\frac{p}{p+2}} \lambda^{\frac{2}{p+2}} \qquad \text{where } C_7 = C_5 + C_6$$

b) Trial wave functions must be orthogonal to the <u>true</u> g.s. wave function. This is most easily accomplished by choosing <u>odd</u> trial wave functions.

For example, $\varphi(x) = x\, e^{-x^2/a^2}$

11. a) Label states $|n_x, n_y\rangle$

ground state: $|0,0\rangle$

$$E^0_{0,0} = \frac{1}{2}\hbar\omega + \frac{1}{2}\hbar\omega = \hbar\omega$$

$$E^1_{0,0} = \langle 0,0| H' |0,0\rangle$$

$$= \langle 0,0| K p_x p_y |0,0\rangle$$

$$p_x = \sqrt{\frac{m\omega\hbar}{2}}\, i(a_x^\dagger - a_x) \qquad p_y = \sqrt{\frac{m\omega\hbar}{2}}\, i(a_y^\dagger - a_y)$$

$$\therefore\ H' = -\frac{K m\omega\hbar}{2}(a_x^\dagger - a_x)(a_y^\dagger - a_y)$$

$$\therefore\ E^1_{0,0} = 0$$

$$E^2_{0,0} = \sum_{\substack{(n_x,n_y)\\ \neq (0,0)}} \frac{|\langle n_x,n_y| H' |0,0\rangle|^2}{E^0_{0,0} - E^0_{n_x,n_y}}$$

$$H'|0,0\rangle = -\frac{K m\omega\hbar}{2}|1,1\rangle$$

$$\therefore\ E^2_{0,0} = \frac{|\langle 1,1| H' |0,0\rangle|^2}{E^0_{0,0} - E^0_{1,1}} = \frac{\left(\frac{K m\omega\hbar}{2}\right)^2}{\hbar\omega - 3\hbar\omega} = -\frac{K^2 m^2 \hbar\omega}{8}$$

$\therefore$ To second order, $E_{0,0} = \hbar\omega - \dfrac{K^2 m^2 \hbar\omega}{8}$

11b) There are two states with unperturbed energy $2\hbar\omega$: $|0,1\rangle$ & $|1,0\rangle$

Need H' as a 2×2 matrix in this subspace.

$$\langle 0,1|H'|0,1\rangle = \langle 1,0|H'|1,0\rangle = 0$$

$$\langle 0,1|H'|1,0\rangle = -\frac{Km\omega\hbar}{2}\langle 0,1|(a_x^\dagger - a_x)(a_y^\dagger - a_y)|1,0\rangle$$

$$\cancel{= -\frac{Km\omega\hbar}{2}\langle 0,1|-a_x^\dagger a_y|}$$

$$= -\frac{Km\omega\hbar}{2}\langle 0,1|-a_x^\dagger a_y|1,0\rangle$$

$$= \frac{Km\omega\hbar}{2}$$

$\therefore$ In $\{|1,0\rangle, |0,1\rangle\}$ subspace,

$$H' = \begin{pmatrix} 0 & Km\omega\hbar/2 \\ Km\omega\hbar/2 & 0 \end{pmatrix}$$

$\therefore$ 1st order corrections $= \pm\, Km\omega\hbar/2$

$\therefore$ To 1st order, energies $= 2\hbar\omega \pm \dfrac{Km\omega\hbar}{2}$

11c) $P_y^3 = -i\left(\frac{m\omega\hbar}{2}\right)^{3/2} (a_y^+ - a_y)^3$

Excitation possible only to those states $|n_x, n_y\rangle$ for which ~~Ω~~

$\langle n_x, n_y | P_y^3 | 0,0\rangle \neq 0$

$a_y a_y^+ a_y^+$
$a_y^+ a_y a_y^+$

$P_y^3 |0,0\rangle = \sqrt{6}\,|3,0\rangle - |1,0\rangle - 2|1,0\rangle$

$= \sqrt{6}\,|3,0\rangle - 5|1,0\rangle$

$\therefore$ Only states that can be excited are $|3,0\rangle$ & $|1,0\rangle$

$|3,0\rangle$: requires $\Omega = 3\omega$

$|1,0\rangle$: " $\Omega = \omega$.

12 a) $-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}u + V(r)\,u = E\,u$

$$V = -\frac{\hbar^2}{2m}\,U \qquad E = \frac{\hbar^2}{2m}k^2$$

so: $-\frac{d^2}{dr^2}u - U(r)\,u = k^2 u$

either form is fine for full credit.

$$u(0) = 0$$

b) WKB wave function:

$$u(r) \propto \sin\left(\frac{1}{\hbar}\int_0^r \sqrt{2m(E - V(r'))}\,dr'\right)$$

sin, to enforce $u(0) = 0$

$$\therefore\ u(r) \propto \sin\left(\int_0^r \sqrt{k^2 + U(r')}\,dr'\right)$$

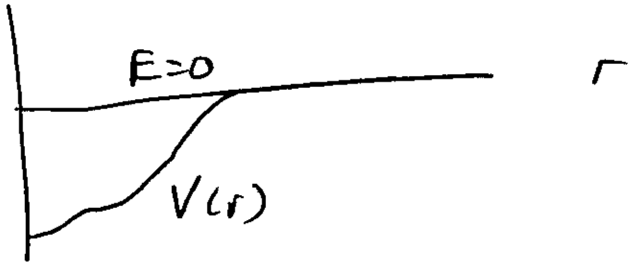
$$\therefore\ u(r) \propto \sin\left(kr + \int_0^r \left(\sqrt{k^2 + U(r')} - k\right)dr'\right)$$

$$\therefore\ \delta_0(k) = \int_0^\infty \left(\sqrt{k^2 + U(r')} - k\right)dr'$$

c) $\lim_{k\to 0} \delta_0^{WKB}(k) = \int_0^\infty \sqrt{U(r')}\,dr'$

This is <u>not</u> $\delta_0 = -ka$ for some a.

What has gone wrong? $\longrightarrow$



WKB solutions not valid where E near V.
(ie turning points). Here, since E = 0
E is near V for all large r !
∴ WKB solution not valid in E→0 limit.

d) $\lim_{k\to\infty} \delta_0^{WKB}(k) = \int_0^\infty \left(k\left(1+\frac{U(r')}{2k^2}+\cdots\right) - k\right) dr'$

$$= \frac{1}{2k}\int_0^\infty U(r')\, dr'$$

Expect this to be a good approximation,
since WKB valid for large $(E-V)$.
Also, makes sense that $\delta \to 0$ for $k\to\infty$.
(Effect of V on very high E wave → 0.)

OPTIONAL, FOR EXTRA CREDIT

Upon ~~spherical~~ angular averaging, only the s-wave contribution to f survives.

$$\therefore \quad f(\theta) = \frac{e^{2i\delta_0} - 1}{2ik}$$

In large-k limit, $\delta_0 \to 0$. $\therefore \ e^{2i\delta_0} \simeq 1 + 2i\delta_0$

$\therefore$ Large-k, ~~spherical~~ angular avg $f = \dfrac{2i\delta_0}{2ik} = \dfrac{\delta_0}{k}$.

So, we need to find large-k, ~~spherical~~ angular avg f^{Born}, and multiply by k. That's ~~δ_0~~ δ_0^{BORN}

$$\lim_{k \to \infty} \delta_0^{\text{BORN}}$$

$$f^{\text{BORN}} = -\frac{m}{2\pi\hbar^2} \int d^3r \, e^{-i\vec{q}\cdot\vec{r}} V(\vec{r})$$

$$= \frac{1}{4\pi} \int d^3r \, e^{-i\vec{q}\cdot\vec{r}} U(\vec{r})$$

$$= \frac{1}{2} \int r^2 dr \int_{-1}^{1} d\cos\theta \, e^{-iqr\cos\theta} U(r)$$

$$= \frac{1}{2} \int r^2 dr \, \frac{e^{-iqr} - e^{iqr}}{-iqr} U(r)$$

$$= \int r^2 dr \, \frac{\sin qr}{qr} U(r)$$

$$= \int r dr \, \frac{\sin qr}{q} U(r)$$

To do the angular average, use:

$$q = 2k \sin\frac{\theta}{2}$$

$$\therefore\ dq = 2k \cos\frac{\theta}{2}\,\frac{d\theta}{2} = -k\cos\frac{\theta}{2}\,\frac{d\cos\theta}{\sin\theta}$$

$$= -\frac{1}{2}k\,\frac{\cos\theta/2}{\cos\theta/2\,\sin\theta/2}\,d\cos\theta$$

$$= -\frac{k^2}{q}\,d\cos\theta$$

$$\therefore\ d\cos\theta = -\frac{q\,dq}{k^2}$$

$$f^{BORN,\,ang.avg} = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, f^{BORN}$$

$$= -\frac{1}{2k^2}\int_{2k}^{0} q\,dq \int r\,dr\,\frac{\sin qr}{q}\,U(r)$$

$$= +\frac{1}{2k^2}\int r\,dr\,U(r)\int_0^{2k} \cancel{q}\,\frac{\sin qr}{\cancel{q}}\,dq$$

$$= -\frac{1}{2k^2}\int r\,dr\,U(r)\,\frac{1}{r}\left(\cos 2kr - 1\right)$$

In large k limit, $\int U(r)\cos 2kr\,dr \to 0$ is highly oscillatory.

because U is smooth & $\cos 2kr$

$$\therefore\ \lim_{k\to\infty} f^{BORN,\,ang.avg} = +\frac{1}{2k^2}\int_0^{\infty} dr\,U(r)$$

$$\therefore\ \lim_{k\to\infty} \delta_0^{BORN} = \frac{1}{2k}\int_0^{\infty} dr\,U(r)$$ Agrees with WKB!