

Quantum Physics III (8.06) Spring 2006

Final Exam Solution

May 18, 2006

1. True or False (20 points):

- (a) (2 points) False
- (b) (2 points) True
- (c) (2 points) False
- (d) (2 points) False
- (e) (2 points) False
- (f) (2 points) True
- (g) (2 points) False
- (h) (2 points) True
- (i) (2 points) True
- (j) (2 points) True

2. Short answer questions (10 points)

- (a) (3 points)

$$\frac{d\sigma}{d\Omega} = \frac{\left(\frac{dN}{dt d\Omega}\right)_{out}}{\left(\frac{dN}{dt DA}\right)_{in}}$$

or

$$\frac{d\sigma}{d\Omega} = \frac{\text{number of particles per unit time per unit solid angle in the detector}}{\text{incoming flux}}$$

- (b) (2 points) weak potential (or more explicitly, the norm of the first order wave function should be much smaller than 1).
- (c) (5 points) First solve the Schrodinger equation for the electron with R fixed. After solving the electron energy spectrum $\mathcal{E}_n(R)$, then in the adiabatic limit, the Hamiltonian for the nuclei becomes

$$H = -\frac{\hbar^2}{2M_N} \nabla_R^2 + V_N(R) + \mathcal{E}_n(R) \quad (1)$$

One then solves the wave functions and spectrum of the nuclei using (1).

3. The Quantum Dipole-Dipole Interaction (10 points)

- (a) (2 points) Note that in natural units e is dimensionless. Since eEL has dimension of energy, we have

$$[E] = eV^2 \quad (2)$$

Since eE has dimension of force we have

$$[F] = (eV)^2 \quad (3)$$

- (b) (3 points) The dipole d has units

$$[d] = eV^{-1}$$

which along with (2) leads to

$$[\alpha] = (eV)^{-3} \quad (4)$$

- (c) (5 points) From

$$|\vec{F}| = k \frac{\alpha_1 \alpha_2}{R^p}$$

and equations (4) and (3) we find that (note that k is dimensionless)

$$(eV)^2 = (eV)^p (eV)^{-6}$$

which leads to

$$p = 8$$

4. A Particle in a Weak Electric Field (22 points)

- (a) (5 points) From the Bohr-Sommerfeld quantization condition (for sharp walls on both sides)

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \sqrt{2m(E + V_0)} = n\pi\hbar, \quad n = 1, 2, \dots \quad (5)$$

The integration is trivial and leads to

$$\sqrt{2m(E_n + V_0)}a = n\pi\hbar \quad (6)$$

and

$$E_n = -V_0 + \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, \dots \quad (7)$$

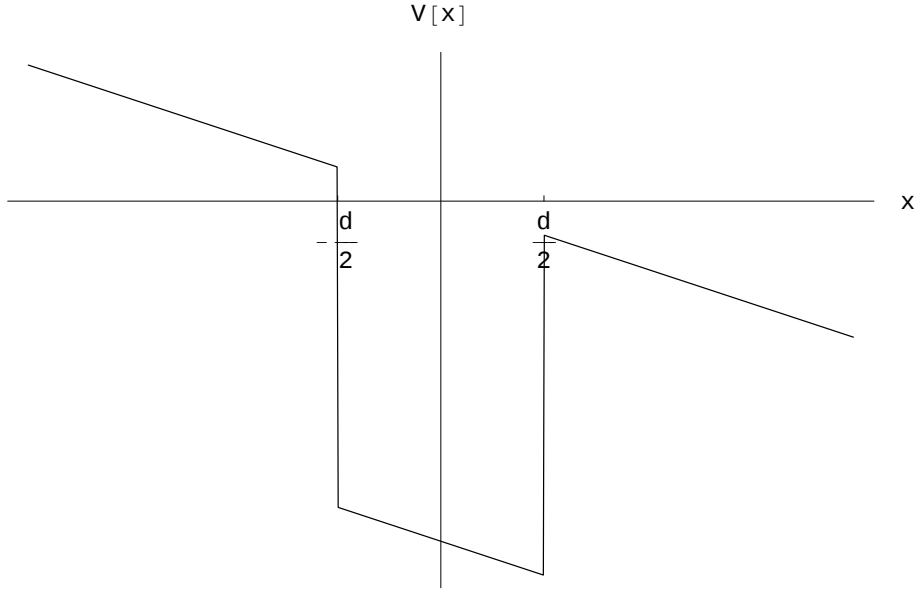


Figure 1: A square well potential in a constant electric field.

- (b) (3 points) Boundary states have energy $E < 0$. Setting $E = 0$ in (29) we find that

$$n = \sqrt{\frac{2ma^2V_0}{\pi^2\hbar^2}}$$

The number of boundary states of the system is thus given by

$$N = \frac{\sqrt{2mV_0}a}{\pi\hbar}$$

- (c) (3 points) Turning on the electric field add an additional term $-e\mathcal{E}_0x$ to the potential. A sketch of the potential is in figure 1.

The potential no longer binds states because it is unbounded from below; a particle trapped in the well can reduce its energy by tunnelling out to $x = \infty$.

- (d) (6 points) Since $V_0 \gg \frac{\hbar^2\pi^2}{ma^2}$ and $V_0 \gg e\mathcal{E}_0a$, we can approximate the ground state energy simply by $-V_0$. The classical turning is

$$x_0 = \frac{V_0}{e\mathcal{E}_0}$$

The barrier penetration factor is given semi-classically by

$$P \sim e^{-S}$$

with

$$S = \frac{2}{\hbar} \int_{\frac{a}{2}}^{x_0} dx \kappa(x) \quad (8)$$

where

$$\kappa(x) = \sqrt{2m(V_0 - e\mathcal{E}_0 x)}$$

Since $V_0 \gg e\mathcal{E}_0 a$ which implies that $x_0 \gg a$, we can set the lower integration limit in (8) to 0 at the lowest order approximation. Then we find that

$$S \approx \frac{4}{3\hbar} \frac{\sqrt{2mV_0^3}}{e\mathcal{E}_0}$$

Thus

$$P \simeq \exp\left(-\frac{4}{3\hbar} \frac{\sqrt{2mV_0^3}}{e\mathcal{E}_0}\right) \quad (9)$$

(e) (5 points) The lifetime τ of the particle is given by

$$\tau = \frac{1}{R}$$

where R is the tunnelling rate, i.e. tunnelling probability per unit time. R can be estimated by

$$R = \frac{P}{T} \quad (10)$$

where P is given by (9) and T is the classical period of the particle trapped in the well. The ground state energy is given by taking $n = 1$ in (29) and we find that

$$E_1 = V_0 + \frac{\pi^2 \hbar^2}{2ma^2}$$

where the second term should be interpreted as the kinetic energy. The velocity v of a classical particle with this energy is thus given by

$$\frac{1}{2}mv^2 = \frac{\pi^2 \hbar^2}{2ma^2}$$

which leads to

$$v = \frac{\pi \hbar}{ma} \quad (11)$$

From (11),

$$T = \frac{2a}{v} = \frac{2ma^2}{\pi \hbar}$$

Thus the lifetime

$$\tau = \frac{2ma^2}{\pi \hbar} e^{+\frac{4}{3\hbar} \frac{\sqrt{2mV_0^3}}{e\mathcal{E}}}$$

5. Derivation of band gap from perturbation theory (15 points)

A particle of mass m moves in one dimension in the periodic potential

$$V(x) = V_0 \cos\left(\frac{2\pi x}{a}\right)$$

- (a) (2 points) The Bloch theorem says that the wave function of the particle can be written in a form

$$\psi(x) = e^{ikx} u_k(x)$$

with $u_k(x)$ a periodic function with period a and $k \in (-\frac{\pi}{a}, \frac{\pi}{a})$. An equivalent expression is

$$\psi(x+a) = e^{ika} \psi(x)$$

- (b) (4 points) Consider normalized plane waves

$$\phi(x) = \frac{1}{\sqrt{L}} e^{ikx}$$

The condition

$$\phi(x+a) = -\phi(x)$$

implies that $e^{ika} = -1$, i.e.

$$k = \frac{(2n+1)\pi}{a}, \quad n = 0, \pm 1, \pm 2, \dots$$

Thus

$$E_k = \frac{(2n+1)^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 0, \pm 1, \pm 2, \dots$$

- (c) (8 points) The lowest energy states in (b) are given by $n = 0$ and $n = -1$, with

$$E_{\pm \frac{\pi}{a}} = \frac{\pi^2 \hbar^2}{2ma^2}$$

and wave functions

$$\psi_{\pm}(x) = \frac{1}{\sqrt{L}} e^{\pm i \frac{\pi}{a} x}$$

Thus we need to consider the degenerate perturbation theory. For this purpose we need to compute $H' = \langle \epsilon | V | \epsilon' \rangle$ with $\epsilon, \epsilon' = \pm$,

$$H'_{++} = H'_{--} = \frac{V_0}{L} \int_0^L dx \cos\left(\frac{2\pi x}{a}\right) = 0$$

and

$$H'_{+-} = H'_{-+} = \frac{V_0}{L} \int_0^L dx e^{i \frac{2\pi x}{a}} \cos\left(\frac{2\pi x}{a}\right) = \frac{V_0}{2}$$

Thus we find that

$$H' = \begin{pmatrix} 0 & \frac{1}{2}V_0 \\ \frac{1}{2}V_0 & 0 \end{pmatrix}$$

We thus find that the lowest energy eigenvalues considered in (b) split into

$$E = \frac{\pi^2 \hbar^2}{2ma^2} \pm \frac{1}{2}V_0 \quad (12)$$

- (d) (1 point) The split V_0 found in (12) is the band gap at $k = \pm \frac{\pi}{a}$ caused by the periodic potential.

6. A slow field reversal (20 points)

The Hamiltonian for an electron in a magnetic field can be written as

$$H = \frac{e}{m} \vec{S} \cdot \vec{B} = \mu_B \begin{pmatrix} B_z & B_x + iB_y \\ B_x - iB_y & -B_z \end{pmatrix}, \quad \mu_B = \frac{e\hbar}{2m} \quad (13)$$

- (a) (3 points) If he reverses the field, there will be a time that the magnetic field is zero, at which there is a degeneracy between spin up and spin down states (w.r.t z -direction) and the adiabatic condition breaks down.
- (b) (3 points) When there is a permanent magnetic field in the $x - y$ plane, then the spin up and spin down states are never degenerate. As far as he reverses the magnetic field slowly enough, the adiabatic condition can be satisfied.
- (c) (4 points) Using (13) the Hamiltonian can be written as

$$H = \begin{pmatrix} vt & \Delta \\ \Delta & -vt \end{pmatrix}$$

with

$$v = \frac{\mu_B B_0}{T}, \quad \Delta = \mu_B B_1$$

- (d) (5 points) In order for the adiabatic condition to be satisfied, we require that the energy uncertainty is not big enough to cause transition to the other level. We thus have

$$\Delta \gg \delta E$$

while

$$\delta E \sim \frac{\hbar}{\delta t} \sim \frac{v\hbar}{\Delta}, \quad \delta t = \frac{\Delta}{v}$$

which leads to

$$\frac{\Delta^2}{\hbar v} \gg 1$$

which is

$$\frac{\mu_B B_1^2 T}{\hbar B_0} \gg 1 \quad \text{or} \quad T \gg \frac{\hbar B_0}{\mu_B B_1^2}$$

- (e) (5 points) For B to rotate the full circle with magnitude B_0 fixed. The energy of the state which aligns with the rotating magnetic field is also constant given by $\mu_B B_0$.

The dynamical phase is

$$\theta = \frac{1}{\hbar} \mu_B B_0 t$$

where t (it is ok if you assume it is $2T$) is the time used to rotate the magnetic field. The Berry's phase is

$$\gamma_+ = -\frac{1}{2} \Omega_c = -\pi$$

7. Born approximation (12 points)

Consider scattering a particle of mass m off a delta-shell potential in 3d

$$V(r) = V_0 b \delta(r - b) \quad (14)$$

- (a) (5 points) Using the formula for Born approximation in a central potential

$$f(\theta) = -\frac{2m}{\hbar^2 q} \int_0^\infty dr r V(r) \sin qr$$

with

$$q = 2k \sin \frac{\theta}{2}$$

one finds that for potential (14)

$$f(\theta) = -\frac{2mV_0 b^2}{\hbar^2 q} \sin qb$$

- (b) (3 points) The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0 b^2}{\hbar^2 q} \right)^2 \sin^2 qb \quad (15)$$

- (c) (4 points) In $k \rightarrow 0$ limit, equation (15) becomes

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0 b^3}{\hbar^2} \right)^2 \quad (16)$$

Thus we find the total cross section

$$\sigma_{tot} = \int d\Omega \frac{d\sigma}{d\Omega} = 4\pi \left(\frac{2mV_0 b^3}{\hbar^2} \right)^2$$

8. Forced harmonic oscillator (15 points)

Consider an harmonic oscillator perturbed under a time dependent external potential for $t \geq 0$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + V(x)f(t)$$

The system is at ground state $|0\rangle$ at $t = 0$.

(a) (9 points) The transition probability is given by

$$P_{a \rightarrow b}(t) = |c_b|^2$$

with

$$c_b(t) = \frac{1}{i\hbar} \int_0^t dt' H'_{ba}(t') e^{i\omega_{ba}t'}, \quad \omega_{ba} = \frac{E_b - E_a}{\hbar}$$

Here $a = 0$ and $b = n > 0$

$$H'_{n0} = \lambda f(t) \langle n|x|0\rangle = K f(t) \delta_{n,1}, \quad K = \lambda \sqrt{\frac{\hbar}{2m\omega}}$$

Thus

$$c_1(\infty) = \frac{K}{i\hbar} \frac{1}{\mu - i\omega} \quad c_n = 0, \quad n > 1$$

and

$$P_{0 \rightarrow 1}(t = \infty) = \frac{K^2}{\hbar^2} \frac{1}{\mu^2 + \omega^2} = \frac{\lambda^2}{2m\hbar\omega} \frac{1}{\mu^2 + \omega^2}$$

(b) (6 points) The system can jump to $|n\rangle$ only if

$$V_n = \langle n|x^8|0\rangle \neq 0$$

and $\Omega = n\omega$. V_n is nonzero for $n = 2, 4, 6, 8$. Thus the values of $\Omega = 2\omega, 4\omega, 6\omega, 8\omega$.

9. Two variational proofs (20 points)

(a) (8 points) Consider a Hamiltonian of the form

$$H = H_0 + \lambda V$$

where the ground state wave function and eigenvalue of H_0 are given by $|\psi_0\rangle$ and E_0 . Then the ground state energy of H from 1st order perturbation theory is given by

$$E = E_0 + E_0^{(1)} = E_0 + \lambda \langle \psi_0 | V | \psi_0 \rangle = \langle \psi_0 | H | \psi_0 \rangle \quad (17)$$

Equation (17) can also be interpreted as a variational computation of the ground state energy with ψ_0 as the trial wave function. Then from the variational principle

$$E \geq E_{gs} \quad (18)$$

where E_{gs} denotes the exact ground state energy of H . Since

$$E_{gs} = E_0 + E_0^{(1)} + E_0^{(2)} + \dots$$

Equation (18) implies that

$$E_0^{(2)} + \dots \leq 0$$

(b) Consider the normalized trial wave function

$$\psi = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}\alpha^2 x^2\right)$$

We want to compute

$$E = \langle \psi | H | \psi \rangle = KE + PE$$

where the kinetic energy is

$$KE = \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx \left| \frac{d\psi}{dx} \right|^2 \quad (19)$$

$$= \frac{\hbar^2 \alpha^2}{4m}. \quad (20)$$

The potential energy is

$$PE = \frac{\alpha}{\sqrt{\pi}} \int_{-a}^a dx V(x) e^{-\alpha^2 x^2} \quad (21)$$

Now let us consider the limit $\alpha \rightarrow 0$. Then (21) becomes

$$PE = \frac{\alpha}{\sqrt{\pi}} \left(\int_{-a}^a dx V(x) + O(\alpha^2) \right) = \alpha V_0 + O(\alpha^3) \quad (22)$$

where we have Taylor expanded $e^{-\alpha^2 x^2}$ inside the integrand of (21) and

$$V_0 = \frac{1}{\sqrt{\pi}} \int_{-a}^a dx V(x) < 0$$

Collecting (20) and (22) we find that as $\alpha \rightarrow 0$

$$E \approx \frac{\hbar^2 \alpha^2}{4m} + \alpha V_0 + O(\alpha^3) \quad (23)$$

Note that in the $\alpha \rightarrow 0$ limit the leading term in (23) is given by the second term which is negative. Therefore there exists a α small enough such that the total energy is less than 0. Hence, the ground state has negative energy, and is a bound state.

10. **Boundary conditions for scattering problems (20 points)**

In three dimensions, the wave function $\psi(\vec{r})$ of a particle scattering off a potential $V(\vec{r})$ should satisfy the boundary condition

$$\psi(\vec{r}) \rightarrow e^{ikz} + \frac{e^{ikr}}{r} f(\theta, \phi), \quad r \rightarrow \infty \quad (24)$$

The first term in (24) is the incoming plane wave along z -direction and the second term is the scattered spherical wave. (r, θ, ϕ) are spherical coordinates with the scattering center located at $r = 0$ and θ the angle between $\vec{r}$ and z -axis.

- (a) (4 points) If the scattering potential $V(\vec{r})$ is central, i.e. $V(\vec{r}) = V(r)$, the system is rotational invariant in ϕ direction. Thus $f(\theta, \phi)$ cannot depend ϕ . The same thing cannot be said of the θ -dependence, since the incoming plane wave breaks the rotational symmetry in θ direction.
- (b) (2 points) In the limit $k \rightarrow 0$, the incoming wave function $e^{ikz} \rightarrow 1$ becomes spherically symmetric. In this limit $f(\theta, \phi)$ does not depend on either θ or ϕ .
- (c) (4 points) In one dimension we should impose the boundary condition

$$\psi(x) = e^{ikx} + Re^{-ikx}, \quad x \rightarrow -\infty \quad (25)$$

$$= Te^{ikx}, \quad x \rightarrow +\infty \quad (26)$$

- (d) (10 points) The boundary condition in 2d should have the form

$$\psi(x, y) = e^{ikx} + \frac{1}{\rho^a} f(\theta), \quad \rho \rightarrow \infty \quad (27)$$

where (ρ, θ) are polar coordinates. This follows from that the scattered wave should be a spherical wave in 2d. The power a in (27) can be determined by requiring that the second term in (27) satisfies the free Schrodinger equation for $\rho \rightarrow \infty$. Free Schrodinger equation in 2d

$$(\nabla^2 + k^2)\psi = 0$$

can be written in polar coordinates as (see equation (9) in the formula sheet)

$$\frac{1}{\sqrt{\rho}} \partial_\rho^2 (\sqrt{\rho} \psi) + \frac{1}{4\rho^2} \psi + \frac{1}{\rho^2} \partial_\phi^2 \psi + k^2 \psi = 0$$

In the limit $\rho \rightarrow \infty$, the above equation becomes

$$\partial_\rho^2 (\sqrt{\rho} \psi) + k^2 \sqrt{\rho} \psi = 0$$

which leads to that

$$\psi(\rho) = \frac{1}{\sqrt{\rho}} f(\theta)$$

We thus in 2d

$$\psi(x, y) = e^{ikx} + \frac{1}{\sqrt{\rho}} f(\theta), \quad \rho \rightarrow \infty \quad (28)$$

Alternatively one can also argue $a = \frac{1}{2}$ in (27) from the conservation of probability.

11. The Landau Problem Revisited (16 points)

(a) (8 points) Choose the gauge,

$$\vec{A} = (0, Bx, 0)$$

and the time-independent Schrodinger equation in this gauge is given by

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2m} \left(p_y - \frac{q}{c} Bx \right)^2$$

$$[p_y, H] = 0$$

we can write the wave function as

$$\psi(x, y) = e^{ik_y y} f(x)$$

Acting on such a wave function the Hamiltonian becomes

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2} m \omega_L^2 (x - x_0)^2$$

with

$$\omega_L = \frac{qB}{mc}, \quad x_0 = \frac{c\hbar k_y}{qB}$$

Thus the energy spectrum is given by

$$E_n = \left(n + \frac{1}{2}\right) \hbar \omega_L, \quad n = 0, 1, \dots \quad (29)$$

(b) (8 points) Expand $\psi(x, y)$ in terms of a complete sets of energy eigenstates

$$\psi(x, y) = \sum_{\alpha} c_{\alpha} \psi_{\alpha}$$

where $\sum_{\alpha}$ is a short hand notion for summing over n and the integral over k_y . Then at time T we find that

$$\Psi(x, y, t) = \sum_{\alpha} c_{\alpha} e^{-\frac{i}{\hbar} E_{\alpha} t} \psi_{\alpha}$$

with E_n given by (29). Now let $t = lT$ with l an integer and $T = \frac{2\pi}{\omega_L}$, we then find that

$$\Psi(x, y; lT) = (-1)^l \psi(x, y)$$

Thus up to an irrelevant overall phase factor $(-1)^l$ (which is universal for all wave functions) $\Psi(x, y; t)$ is periodic in t with a period T .