

Quantum Physics III (8.06) Spring 2006

MIDTERM TEST

Thursday March 23, 2006

You have 1 hour and 20 minutes.

There are 5 problems, totalling 80 points. Do all problems.

Answer all problems in the white books provided.

Write YOUR NAME on EACH white book you use.

Budget your time wisely, using the point values as a guide. Note that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

For an energy eigenstate ψ of energy E

$$\psi(t) = e^{-\frac{i}{\hbar}Et} \psi(0)$$

and the Schrodinger equation reduces to an eigenvalue equation

$$H\psi = E\psi$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar \omega \left(\hat{N} + \frac{1}{2} \right)$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} + i\hat{p}) , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} - i\hat{p}) ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 .$$

Conversely

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{1}{i} \sqrt{\frac{\hbar m \omega}{2}} (\hat{a} - \hat{a}^\dagger)$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left(-\frac{m\omega}{2\hbar} x^2 \right) .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

- Natural units

In the natural units, the dimension of any physical quantity can be written as

$$(eV)^a \hbar^b c^d$$

It is often convenient to set $c = \hbar = 1$. Then the dimension of any physical quantity can be written in terms of powers of electron Volts. For example

$$[e] = (eV)^0 \text{ (dimensionless)}$$

$$[m] = eV$$

$$[L] = eV^{-1}$$

$$[t] = eV^{-1}$$

- Particle in an Electric and/or Magnetic Field:

The Hamiltonian for a particle with charge q in a magnetic field and electric field

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

is:

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\phi \quad (1)$$

- Gauge invariance:

If $\psi(\vec{x}, t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x}, t) = \exp \left(-\frac{iq}{\hbar c} f(\vec{x}, t) \right) \psi(\vec{x}, t)$$

solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla} f$ and replacing ϕ by $\phi' = \phi + (1/c) \partial f / \partial t$.

- Time independent perturbation theory:

Suppose that

$$H = H_0 + H'$$

where we already know the eigenvalues $E_n^{(0)}$ and eigenstates $|\psi_n^{(0)}\rangle$ of H^0 :

$$H_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle .$$

Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^{(0)} + H'_{nn} + \sum_{m \neq n} \frac{|H'_{nm}|^2}{E_n^{(0)} - E_m^{(0)}} + \dots \quad (2)$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \sum_{m \neq n} \frac{H'_{mn}}{E_n^{(0)} - E_m^{(0)}} |\psi_m^{(0)}\rangle + \dots \quad (3)$$

where $H'_{nm} \equiv \langle \psi_n^{(0)} | H' | \psi_m^{(0)} \rangle$.

If H_0 has degeneracy at $E_n^{(0)}$, first diagonalize H' in the corresponding degenerate subspace, then use equations (2) and (3). In particular $|\psi_n^{(0)}\rangle$ (“good states”) should be one of the eigenvectors of H' in the degenerate subspace.

1. Short Answer (16 points)

- (a) (1 point) Multiple choice: The energy split between levels $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ in a hydrogen atom is an example of
(A) fine structure splitting (B) Lamb shift (C) Zeeman effect
- (b) (2 points) Consider an electron in a strip of conductor of finite size lying in the $x - y$ plane with a constant magnetic field along the z direction. Are the following statements *true or false*:
(i) the degeneracy of a Landau level decreases with increasing magnetic field.
(ii) The separation in energy between Landau levels increases with increasing magnetic field.
- (c) (3 points) The quantum of conductance observed in the Integer Quantum Hall Effect is a combination of fundamental constants. What is it? If you do not remember the answer, you can deduce it by dimensional analysis. Note that the conductance has dimension of velocity. **[No derivation required. And, you need not get the numerical coefficient right to get full credit.]**
- (d) (5 points) Suppose we have noninteracting electrons in a box, with number density n . Describe at a qualitative level how it can be that for certain values of n , if we turn on a weak periodic potential the material becomes an insulator. Your explanation should include words, and may include sketches. There need be no equations, and you need not derive what you describe.
- (e) (5 points) Sketch an experiment to measure the Aharonov-Bohm effect, and comment on the significance of the electromagnetic potentials $\vec{A}$ and ϕ in quantum mechanics as opposed to classical physics.

2. Dimensional Analysis of a Degenerate Fermi Gas (10 points)

Consider a gas of fermions in a cubic box at zero temperature. The system has an *energy density* ϵ (i.e. energy per unit volume) proportional to the number density n (i.e. number of electrons per unit volume) raised to some power γ :

$$\epsilon = K n^\gamma .$$

In general, the constant K may depend on three constants: $\hbar$, c , and the fermion mass m .

- (a) (5 points) For a *non-relativistic* Fermi gas, deduce from physical reasoning how K should depend on m . Then use dimensional analysis to find the numerical value of γ .
- (b) (5 points) For an *ultra-relativistic* Fermi gas, deduce from physical reasoning how K should depend on m . Then use dimensional analysis to find the numerical value of γ .

(Hint: using the natural units could simplify your analysis.)

3. Fermi surface (10 points)

Consider 8 electrons in a one-dimensional harmonic oscillator potential with frequency ω . You should consider the spin of electrons.

- (a) (3 points) Find the ground state energy of the 8-electron system.
- (b) (3 points) Find the energy of the first excited state of the 8-electron system.
- (c) (4 points) Suppose it is 8 bosons rather than electrons, answer (a) and (b).

4. Perturbations of a Two-Dimensional Harmonic Oscillator (26 points)

Consider a quantum system described by the Hamiltonian

$$H = H_0 + H'$$

where H_0 is the two-dimensional harmonic oscillator Hamiltonian

$$H_0 = (a_x^\dagger a_x + a_y^\dagger a_y + 1)\hbar\omega.$$

The energy eigenstates of the system can be labelled by the occupation numbers in x, y directions

$$|n_x, n_y\rangle = \frac{(a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y}}{\sqrt{n_x! n_y!}} |0, 0\rangle$$

The perturbing Hamiltonian H' is given by

$$H' = \Delta(a_x + a_x^\dagger)(a_y + a_y^\dagger),$$

with Δ a constant.

See p.2. for useful formulae of the one-dimensional harmonic oscillator.

- (a) (8 points) Evaluate the first and second order in Δ corrections to the ground state energy.
- (b) (6 points) Write down the unperturbed state(s) whose energy is $2\hbar\omega$. Evaluate the energy of these state(s) to first order in Δ .
- (c) (4 points) Find the correct zero-th order states (“good states”) for the energy levels considered in part (b).
- (d) (6 points) Find the second order corrections in Δ to the energy of the states considered in part (c). **(If there are several states, you only need to compute the correction to one of them.)**
- (e) (2 points) What is the rough range of Δ that you expect the perturbation theory to be a good approximation? **(no explanation or derivation required)**

5. **Particle in a magnetic field and harmonic oscillator potential (18 points)**

A particle of charge q and mass m moves in a uniform magnetic field B directed along the z -axis. The particle also experiences a harmonic oscillator potential given by

$$V(x) = \frac{1}{2}m\omega_0^2x^2 \quad (4)$$

We will only consider the motion in the $x - y$ plane.

- (a) (5 points) Choose an appropriate gauge and write down the Hamiltonian in that gauge. State your reasoning for choosing the gauge. [Hint: You will get full credit for this part for any gauge you choose. However it would be desirable if you choose a gauge which is the simplest to do part (c).]
- (b) (5 points) Derive the energy eigenvalues of the Hamiltonian **in the absence of** of the potential (4) using the gauge you choose in (a).
- (c) (8 points) Find the energy spectrum of the particle **in the presence of** of the potential (4) using the gauge you choose in (a). (For convenience of computation, take $\omega_0 = \omega_L$, with ω_L the cyclotron frequency.)