

Quantum Physics III (8.06) Spring 2004

MIDTERM TEST

Thursday March 18, 2004

You have 1 hour and 20 minutes.

There are 6 problems, totalling 80 points. Do all problems.

Answer all problems in the blue books provided.

Write YOUR NAME on EACH blue book you use.

Budget your time wisely, using the point values as a guide. Note that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(t) |\psi(t)\rangle$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega \left(\hat{N} + \frac{1}{2} \right)$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} + i\hat{p}) , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}} (m\omega\hat{x} - i\hat{p}) ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle , \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp \left(-\frac{m\omega}{2\hbar} x^2 \right) .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp(-ax^2) = \sqrt{\frac{\pi}{a}}$$

- Some useful constants:

$$\hbar c = 197 \times 10^{-7} \text{ eV cm}$$

The mass of the electron is $m_e = 0.511 \text{ MeV}/c^2$. If B is 1 gauss, then the force eB is 300 eV/cm.

- Particle in an Electric and/or Magnetic Field:

The Hamiltonian for a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and an electric field $\vec{E} = -\vec{\nabla}\phi$ is:

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\phi \tag{1}$$

- Gauge invariance:

If $\psi(\vec{x}, t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x}, t) = \exp\left(-\frac{iq}{\hbar c}f(\vec{x}, t)\right) \psi(\vec{x}, t)$$

solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla}f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

Suppose that

$$H = H^0 + H'$$

where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0 :

$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle .$$

Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^0 + H'_{nn} + \sum_{m \neq n} \frac{|H'_{nm}|^2}{E_n^0 - E_m^0} + \dots$$

$$|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m \neq n} \frac{H'_{mn}}{E_n^0 - E_m^0} |\psi_m^0\rangle + \dots$$

where $H'_{nm} \equiv \langle \psi_n^0 | H' | \psi_m^0 \rangle$.

1. Short Answer Questions (8 points)

No derivations required. And, you need not get overall signs or factors of 2π right to get full credit.

- (a) (4 points) Consider an infinite solenoid, containing magnetic flux Φ , and a quantum mechanical “bead on a ring”, where the diameter of the ring is much larger than the thickness of the solenoid.
- How can the ring be positioned relative to the solenoid so as to ensure that the energy eigenvalues of the bead are completely independent of the value of Φ ? If there is no such position, say so.
 - Now, suppose the ring is positioned in some way such that it does not intersect the solenoid, but you do not know anything further about its position. For what values of Φ , if any, can you be confident that the energy eigenvalues of the bead are the same as they would be if Φ were zero?
- (b) (4 points) Consider electrons moving in the (x, y) -plane, in a magnetic field pointing in the z -direction, $\vec{B} = (0, 0, B_0)$. An electric field $\vec{E} = (E_x, E_y)$ is applied in the (x, y) -plane, and the resulting current in the plane $\vec{j} = (j_x, j_y)$ is measured. The 2×2 conductivity matrix σ is defined via $\vec{j} = \sigma \vec{E}$. The strength of the magnetic field B_0 is such that N Landau levels are filled. What is σ ?

2. Casimir Force by Dimensional Analysis (8 points)

Two infinite grounded parallel conducting plates in vacuum separated by a distance d feel an attractive force per unit area (due to quantum fluctuations) that depends only on d and the fundamental constants $\hbar$ and c . Use dimensional analysis to find the dependence of the force per unit area on d , $\hbar$ and c . [That is, the force per unit area is $K d^\alpha \hbar^\beta c^\gamma$, with K a dimensionless constant, and you must derive the values of α , β and γ .]

3. Stark Effect in Hydrogen (10 points)

Consider the first excited state of hydrogen. (The $n = 2$ level.) In this problem, you may neglect all fine and hyperfine structure, and may neglect spin. You need not show any calculations.

- (a) (2 points) List the states in the $n = 2$ level, using the standard $|n\ell m\rangle$ notation.
- (b) (8 points) In the presence of a constant applied electric field in the z -direction, the degeneracy among these states is broken. What are the $n = 2$ eigenstates of the Hamiltonian in the presence of a weak applied electric field? Which of these states, if any, have the same energy as they would in the absence of an electric field?

4. Eigenstates of the “Translation” Operator (12 points)

Consider an infinite dimensional Hilbert space with orthonormal basis states that we will call $|n\rangle$ where n is an integer running from $-\infty$ to $+\infty$.

The Hamiltonian for the system is

$$H = \sum_{n=-\infty}^{n=+\infty} \left[E_0 |n\rangle\langle n| + \Delta |n\rangle\langle n+1| + \Delta |n+1\rangle\langle n| \right].$$

Note that the basis states are not energy eigenstates.

Define the “translation” operator T by:

$$T|n\rangle = |n+1\rangle$$

- (a) (4 points) Show that H and T commute.
- (b) (4 points) Find the state which is an eigenstate of T with eigenvalue $\exp(-i\theta)$. (Call this state $|\theta\rangle$.)
- (c) (4 points) Show that $|\theta\rangle$ is an eigenstate of H . What is its energy?

5. Perturbation of a Two-Dimensional Harmonic Oscillator (22 points)

Consider a quantum system described by the Hamiltonian

$$H = H^0 + H'$$

where H^0 is the two-dimensional harmonic oscillator Hamiltonian

$$H^0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2)$$

and where the perturbing Hamiltonian H' is given by

$$H' = Kxy,$$

with K a constant.

- (a) (12 points) Evaluate the ground state energy to second order in K . [Note: this means evaluate the zeroth, first and second order contributions to the ground state energy.]
- (b) (10 points) Evaluate the energy of the state(s) whose unperturbed energy (ie energy if K were zero) is $2\hbar\omega$ to first order in K .

6. The Landau Problem Revisited (20 points)

Consider a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ described by the Hamiltonian

$$H = \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 . \quad (2)$$

The particle is restricted to move in the (x, y) plane, and the magnetic field is $\vec{B} = (0, 0, B_0)$.

Define the “velocity operator”

$$\vec{v} = \frac{1}{m} \left(\vec{p} - \frac{q}{c} \vec{A} \right) .$$

- (a) (6 points) Explain what it means for an operator to be gauge invariant, and demonstrate that $\vec{v}$ is gauge invariant. [Note: you may use facts about gauge invariance from the information sheet.]
- (b) (7 points) You do not need to have completed part (a) to do parts (b) and (c). Evaluate $[v_x, v_y]$.
- (c) (7 points) Write the Hamiltonian in terms of gauge invariant operators only, and then use the result of (b) to show that this Hamiltonian can be recast as a simple harmonic oscillator, written in terms of gauge invariant operators only. Determine the energy eigenvalues.

[Note: you will get almost no credit for simply writing down the energy eigenvalues. The purpose of this problem is for you to derive the eigenvalues from a gauge invariant starting point.]