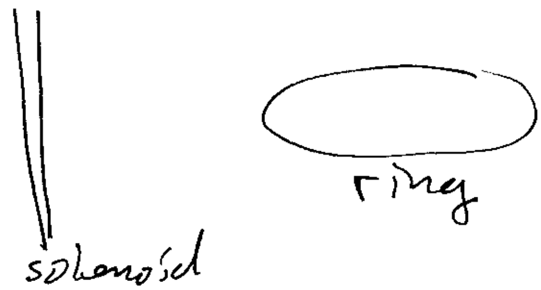
8.06 MIDTERM 3/18/2004

SOLUTION SET

1 a) i)



any position so that the solenoid does not go through the ring.

ii) $\dfrac{Nhc}{e}$ for N any integer

[Full credit if $h \to \hbar$]

b) $$\sigma = \begin{pmatrix} 0 & Ne^2/h \\ -Ne^2/h & 0 \end{pmatrix}$$

[Full credit if $h \to \hbar$ or if overall sign wrong.]

2. $$\left[\frac{F}{A}\right] = \left[\frac{E}{L^3}\right] = \left[\frac{\hbar c}{L^4}\right]$$

$$[d] = [L] = \cancel{\left[\frac{\hbar c}{E}\right]}$$

$$\therefore \quad \frac{F}{A} = K\, d^{-4}\, \hbar^{1}\, c^{1}$$

3 a) $|200\rangle$

$|211\rangle$

$|210\rangle$

$|21\,{-1}\rangle$

b) Need to know which of the 16 matrix elements $\langle n\,l\,m|\,H'\,|n\,\tilde{l}\,\tilde{m}\rangle$, for $|nlm\rangle$, $|n\tilde{l}\tilde{m}\rangle$ any of the 4 states in (a), are nonzero.

Parity $\Rightarrow$ $l=0, \tilde{l}=1$ or $l=1$ $\tilde{l}=0$. $\therefore H' \sim z$ is odd

azimuthal integral $\Rightarrow$ $m=\tilde{m}$.

$\therefore H'$ is indep. of azimuthal angle.

$\therefore$ $\langle 210|H'|200\rangle$ and $\langle 200|H'|210\rangle$ are the only nonzero matrix elements.

So, the answer to the problem is:

<u>Eigenstate</u>

$\left.\begin{array}{l}|211\rangle \\ |21\,{-1}\rangle\end{array}\right\}$ energies <u>are</u> <u>not</u> affected by $\vec{\varepsilon}$.

$\left.\begin{array}{l}|210\rangle+|200\rangle \\ |210\rangle-|210\rangle\end{array}\right\}$ energies <u>are</u> affected by $\vec{\varepsilon}$

[Full credit for correct answer without explanation. If answer incorrect, partial credit ~~based on~~ may depend on content of explanations given.]

4(a) Evaluate $TH|m\rangle$ and $HT|m\rangle$, for arbitrary $|m\rangle$:

$$HT|m\rangle = H|m+1\rangle = E_0|m+1\rangle + \Delta|m\rangle + \Delta|m+2\rangle$$

$$TH|m\rangle = T\left\{E_0|m\rangle + \Delta|m-1\rangle + \Delta|m+1\rangle\right\}$$

$$= E_0|m+1\rangle + \Delta|m\rangle + \Delta|m+2\rangle$$

$\therefore$ $[H,T]=0$.

b) $$|\theta\rangle = \sum_n e^{in\theta}|n\rangle$$

$$T|\theta\rangle = \sum_n e^{in\theta}|n+1\rangle$$

$$= \sum_m e^{im\theta - i\theta}|m\rangle \qquad [m = n+1]$$

$$= e^{-i\theta}\sum_m e^{im\theta}|m\rangle$$

$$= e^{-i\theta}|\theta\rangle.$$

$\therefore$ $|\theta\rangle$ is eigenstate of T with eigenvalue $e^{-i\theta}$

c) $$H|\theta\rangle = \sum_n\left[E_0|n\rangle\langle n| + \Delta|n\rangle\langle n+1| + \Delta|n+1\rangle\langle n|\right] \times \sum_m e^{im\theta}|m\rangle$$

$$= \sum_m\left\{E_0 e^{im\theta}|m\rangle + \Delta e^{im\theta}|m-1\rangle + \Delta e^{im\theta}|m+1\rangle\right\}$$

$$= \sum_n\left\{E_0 e^{in\theta}|n\rangle + \Delta e^{i\theta}e^{in\theta}|n\rangle + \Delta e^{-i\theta}e^{in\theta}|n\rangle\right\}$$

$$= \left[E_0 + \Delta e^{i\theta} + \Delta e^{-i\theta}\right]\sum_n e^{in\theta}|n\rangle = \left[E_0 + 2\Delta\cos\theta\right]|\theta\rangle$$

$\therefore$ $|\theta\rangle$ is eigenstate of H with eigenvalue $E_0 + \cancel{2\Delta\cos\theta}$ $2\Delta\cos\theta$.

5 a) Denote states by $|n_x, n_y\rangle$

with unperturbed energy $\hbar\omega(n_x + n_y + 1)$

$$E_0 = E_0^0 + E_0^1 + E_0^2 + \cdots$$

where: $E_0^0 = \hbar\omega$

$$E_0^1 = \langle 0,0 | H' | 0,0\rangle$$

$$= \frac{K\hbar}{2m\omega} \langle 0,0 | (a_x + a_x^\dagger)(a_y + a_y^\dagger) | 0,0\rangle$$

$$= 0$$

$$\cancel{E_0^2 = \sum_{\substack{|m,n\rangle \\ \neq |0,0\rangle}} \frac{|\langle m,n | H' | 0,0\rangle|^2}{\hbar\omega - \hbar\omega(}}$$

$$E_0^2 = \sum_{\substack{|n_x, n_y\rangle \\ \neq |0,0\rangle}} \frac{|\langle n_x, n_y | H' | 0,0\rangle|^2}{\hbar\omega - \hbar\omega(n_x + n_y + 1)}$$

$$H'|0,0\rangle = \frac{K\hbar}{2m\omega} (a_x + a_x^\dagger)(a_y + a_y^\dagger)|0,0\rangle$$

$$= \frac{K\hbar}{2m\omega} |1,1\rangle$$

$\therefore$ in the sum, only $n_x = n_y = 1$ gives nonzero H' matrix element.

$$\therefore E_0^2 = \frac{|\langle 1,1 | H' | 0,0\rangle|^2}{\hbar\omega - 3\hbar\omega} = \frac{(K\hbar/2m\omega)^2}{-2\hbar\omega} = -\frac{K^2\hbar}{8m^2\omega^3}$$

$\therefore E_{\text{ground state}} = \hbar\omega - \frac{K^2\hbar}{8m^2\omega^3}$, to order K^2

5b) The unperturbed states with unperturbed energy is $2\hbar\omega$ are: $|0,1\rangle$ and $|1,0\rangle$

$$\langle \underset{n_x}{1},0 | H' | \underset{n_x}{1},0\rangle = \langle 0,1 | H' | 0,1\rangle = 0$$

$$\langle 1,0 | H' | 0,1\rangle = \frac{K\hbar}{2m\omega} \langle 1,0 | (a_x + a_x^+)(a_y + a_y^+) | 0,1\rangle$$

$$= \frac{K\hbar}{2m\omega}$$

$\therefore$ in this degenerate subspace,

$$H' = \begin{pmatrix} 0 & K\hbar/2m\omega \\ K\hbar/2m\omega & 0 \end{pmatrix}$$

$\therefore$ The good eigenstates are $\left.\begin{array}{r} |1,0\rangle + |0,1\rangle \\ \text{and } |1,0\rangle - |0,1\rangle \end{array}\right\}$ not asked for in the problem

and their energies are:

$$2\hbar\omega + \frac{K\hbar}{2m\omega}$$

$$\text{and } 2\hbar\omega - \frac{K\hbar}{2m\omega}$$

6 a) An operator $\mathcal{O}$ is gauge invariant

if $\langle\psi|\mathcal{O}|\psi\rangle = \langle\psi'|\mathcal{O}'|\psi'\rangle$

where: $\mathcal{O}'$ is the gauge transform of $\mathcal{O}$

and $\psi'(\vec{x},t) = e^{-\frac{iq}{\hbar c} f(\vec{x},t)} \psi(\vec{x},t)$

with f the function that specifies the gauge transformation, as on the information sheets.

For $\mathcal{O} = \vec{v}$:

$$\vec{v}' = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}'\right) = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla} f\right)$$

$$\langle\psi|\vec{v}|\psi\rangle = \frac{1}{m}\int d^2x\ \psi^*(\vec{x})\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}\right)\psi(\vec{x})$$

$$\langle\psi'|\vec{v}'|\psi'\rangle = \frac{1}{m}\int d^2x\ \psi'^*\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla} f\right)\psi'$$

$$= \frac{1}{m}\int d^2x\ e^{+\frac{iq}{\hbar c}f}\psi\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A} + \frac{q}{c}\vec{\nabla} f\right)e^{-\frac{iq}{\hbar c}f}\psi$$

$$= \frac{1}{m}\int d^2x\ e^{+\frac{iq}{\hbar c}f}\psi\left(-i\hbar\vec{\nabla}\psi - \frac{q}{c}\vec{A}\psi + \frac{q}{c}(\vec{\nabla} f)\psi - \frac{q}{c}(\vec{\nabla} f)\psi\right) \times e^{-\frac{iq}{\hbar c}f}$$

$$= \langle\psi|\vec{v}|\psi\rangle$$

6 b) $[V_x, V_y] = \frac{1}{m^2}\left[P_x - \frac{q}{c}A_x,\ P_y - \frac{q}{c}A_y \right]$

$$= \frac{1}{m^2}\left(-\frac{q}{c}\right)\left\{ [P_x, A_y] + [A_x, P_y] \right\}$$

$$= -\frac{q}{m^2 c}\left\{ -i\hbar \frac{\partial A_y}{\partial x} + i\hbar \frac{\partial A_x}{\partial y} \right\}$$

$$= i\hbar \frac{q}{m^2 c} B_z$$

$$= i\hbar \frac{q B_0}{m^2 C}$$

c) $H = \frac{1}{2} m \left(V_x^2 + V_y^2 \right)$

A simple harmonic oscillator has

$$H_{SHO} = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

where x, P are some operators satisfying $[x, P] = i\hbar$

So... Define operators $\hat{\theta}, \hat{\Pi}$ via:

$$\hat{\Theta} \equiv \frac{mc}{qB_0} V_x \qquad \hat{\Pi} \equiv m V_y$$

Then, our Hamiltonian becomes

$$H = \frac{1}{2} m \left(V_x^2 + V_y^2 \right) = \frac{1}{2} m \left(\frac{qB_0}{mc} \right)^2 \theta^2 + \frac{\Pi^2}{2m}$$

And... $[\theta, \Pi] = \frac{mc}{qB_0} m [V_x, V_y] = \frac{m^2 c}{qB_0} i\hbar \frac{qB_0}{m^2 c} = i\hbar$

So, our Hamiltonian is a S.H.O. with $\omega = \frac{qB_0}{mc}$.

$\therefore$ Energy eigenvalues are $\hbar \frac{qB_0}{mc}\left(n + \frac{1}{2}\right)$ for $n =$ integer.