Quantum Physics III (8.06) Spring 2005
MIDTERM TEST
Thursday March 17, 2004
You have 1 hour and 20 minutes.

There are 6 problems, totalling 80 points. Do all problems.

Answer all problems in the blue books provided.
Write YOUR NAME on EACH blue book you use.
Budget your time wisely, using the point values as a guide. Note that shorter problems may not always be easier problems.

No books, notes or calculators allowed.

Some potentially useful information

- Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$$

where

$$[\hat{x}, \hat{p}] = i\hbar \ .$$

This Hamiltonian can be rewritten as

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, and the operators $\hat{a}$ and $\hat{a}^\dagger$ are given by

$$\hat{a} = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} + i\hat{p}\right) \ , \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}\left(m\omega\hat{x} - i\hat{p}\right) \ ,$$

and satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \ .$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on eigenstates of $\hat{N}$ is given by

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \ , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \ .$$

The ground state wave function is

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \ .$$

- Gaussian integral

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

- Angular momentum operators in spherical coordinates

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \ ; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle \theta,\phi|\ell,m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \ ; \quad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi) \ ; \quad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi) \ ; \quad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi) \ ;$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

- Some useful constants:

$$\hbar c = 197\times 10^{-7} \text{ eV cm}$$

The mass of the electron is $m_e = 0.511$ MeV$/c^2$. If B is 1 gauss, then the force eB is 300 eV/cm.

- Particle in an Electric and/or Magnetic Field:

The Hamiltonian for a particle with charge q in a magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and an electric field $\vec{E} = -\vec{\nabla}\phi$ is:

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \qquad (1)$$

- Gauge invariance:

If $\psi(\vec{x},t)$ solves the Schrödinger equation defined by the Hamiltonian (1), then

$$\psi'(\vec{x},t) = \exp\left(-\frac{iq}{\hbar c}f(\vec{x},t)\right)\psi(\vec{x},t)$$

solves the Schrödinger equation obtained upon replacing $\vec{A}$ by $\vec{A}' = \vec{A} - \vec{\nabla}f$ and replacing ϕ by $\phi' = \phi + (1/c)\partial f/\partial t$.

- Time independent perturbation theory:

Suppose that

$$H = H^0 + H'$$

where we already know the eigenvalues E_n^0 and eigenstates $|\psi_n^0\rangle$ of H^0:

$$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle \ .$$

Then, the eigenvalues and eigenstates of the full Hamiltonian H are:

$$E_n = E_n^0 + H'_{nn} + \sum_{m\neq n}\frac{|H'_{nm}|^2}{E_n^0 - E_m^0} + \ldots$$

$$|\psi_n\rangle = |\psi_n^0\rangle + \sum_{m\neq n}\frac{H'_{mn}}{E_n^0 - E_m^0}|\psi_m^0\rangle + \ldots$$

where $H'_{nm} \equiv \langle\psi_n^0|H'|\psi_m^0\rangle$.

1. **Short Answer (10 points)**

   (a) (5 points) Suppose we have noninteracting electrons in a box, with number density n. Describe at a qualitative level how it can be that for certain values of n, if we turn on a weak periodic potential the material becomes an insulator. Your explanation should include words, and may include sketches. There need be no equations, and you need not derive what you describe.

   (b) (1 point) If you dump some matter onto a stable neutron star, increasing its mass, does its radius get bigger or smaller? [No explanation required.]

   (c) (4 points) Use the facts about gauge transformations given on the information sheets to show that x is a gauge invariant operator, whereas p is not.

2. **Polarizability of Hydrogen, by Dimensional Analysis (8 points)**

   When a hydrogen atom in its ground state is placed in an electric field $\mathcal{E}$, the electric field distorts the ground state wave function and shifts the ground state energy E_0. The definition of the polarizability α is

   $$\alpha \equiv \frac{\partial^2 E_0}{\partial \mathcal{E}^2}$$

   (a) (2 points) Which of the following parameters and constants of nature could be expected to occur in the expression for α: $\hbar$, c, Newton's constant G_N, e, the electron mass m, the proton mass M? And, which should not occur?

   (b) (6 points) Use dimensional analysis to determine α, leaving undetermined only a purely numerical factor.

3. **Pressure Without Interaction (15 points)**

   (a) (14 points) Consider N noninteracting spin-1/2 fermions in a cubic box, with sides of length L. The fermions have mass m. Assume the fermions motion is nonrelativistic. Evaluate the total kinetic energy E_{TOT} of the N fermions, assuming they are in the lowest energy state. Use the definition $P = -dE_{\text{TOT}}$ to evaluate the pressure of the gas of noninteracting fermions. Write your answer in terms of the number density $n \equiv N/L^3$, and other parameters in the problem.

   (b) (1 point) Evaluate E_{TOT} and P for N noninteracting bosons in a box.

4. **The Landau Problem (20 points)**

Consider an electron which is restricted to move in the xy-plane, and suppose that the sample is a square, with $-L/2 < x < L/2$ and $-L/2 < y < L/2$.

Apply a uniform magnetic field $\vec{B} = (0, 0, B)$ perpendicular to the plane.

(a) (3 points) Write the Hamiltonian H in a gauge in which p_y commutes with the Hamiltonian.

(b) (15 points) Derive the energy eigenvalues, and the degeneracy of each level. [You may assume that the wave functions satisfy periodic boundary conditions in the y-direction.]

(c) (7 points) Now, turn on an electric field in the x direction, $\vec{E} = (E, 0, 0)$. Write the Hamiltonian, in the same gauge that you have been using throughout. Determine the energy eigenvalues.

5. **The Integer Quantum Hall Effect, and the Aharonov-Bohm Effect (12 points)**

Consider a two-dimensional electron gas in an annular-shaped sample, as shown in the diagram below. The entire annulus is immersed in a spatially uniform, time-independent magnetic field with strength B, oriented perpendicular to the plane of the annulus. There is a solenoid inserted through the center of the annulus. Outside the solenoid, the only magnetic field is the static B. Inside the solenoid, there is an additional time-dependent magnetic flux $\Phi(t)$.

The density of electrons in the sample is such that the sample is on the second integer quantum hall effect plateau, with Hall conductance is $\sigma_H = 2e^2/h$. The sample is a real sample, impurities and all.

An ”ammeter” which is sensitive enough to count electron by electron is connected between the inner and outer edge of the annulus.

Derive a relation between $\Phi(t)$ and the number of electrons passing through the ammeter between time 0 and time T. Explain carefully what features of the Aharonov-Bohm effect are important in deriving the relation, and why.

[Note: Faraday's law states that the integral of $\vec{E} \cdot \vec{dl}$ around a closed contour is $-(1/c)d\Phi/dt$. Here, $\Phi(t)$ is the magnetic flux through the contour, and $\vec{E}$ is the electric field at the contour.]

6. **Particle on a Sphere, with Three Different Perturbations (25 points)**

Note: spherical harmonics that you will need for this problem are given on the information sheets. Integrals that you might need are given below.

Consider a particle with mass M restricted to move on a sphere with radius $r = 1$. The wave function for the particle is thus a function only of the angles θ and ϕ.

In the absence of any potential, the Hamiltonian for this system is

$$H^0 = \frac{p^2}{2M} = \frac{L^2}{2Mr^2} = \frac{L^2}{2M} \ .$$

The particle has charge q, and has no spin.

(a) (2 points) What are the energies and degeneracies of the states with angular momentum quantum number $\ell = 0, 1$?

(b) (4 points) Now, turn on a magnetic field in the z-direction $\vec{B} = B_0\hat{z}$. To first order in B_0, the term added to the Hamiltonian as a result of the magnetic field is given by

$$H' = -\frac{q}{2Mc}\vec{B}\cdot\vec{L}$$

Calculate the energies of the $\ell = 1$ state(s) to first order in B_0.

(c) (11 points) Turn the magnetic field from part (b) off. Now, turn on a perturbation described by the Hamiltonian

$$H' = \epsilon xy$$

Calculate the energies of the $\ell = 1$ state(s) to first order in ϵ.

(d) (8 points) Turn off the perturbation from part (c). Now, turn on an electric field in the z-direction, $\vec{E} = (0, 0, E_0)$. Calculate the energies of the $\ell = 0$ state(s) to *second* order in E_0.

Here are some integrals, some of which you might find useful:

$$\begin{aligned}
\int_0^\pi d\theta \sin\theta &= 2 \\
\int_0^\pi d\theta \sin^2\theta &= \frac{\pi}{2} \\
\int_0^\pi d\theta \sin^3\theta &= \frac{4}{3} \\
\int_0^\pi d\theta \sin^4\theta &= \frac{3\pi}{8} \\
\int_0^\pi d\theta \sin^5\theta &= \frac{16}{15} \\
\int_0^\pi d\theta \sin^6\theta &= \frac{5\pi}{16} \\
\int_0^\pi d\theta \sin^n\theta\cos\theta &= 0
\end{aligned}$$

$$
\begin{aligned}
\int_0^\pi d\theta \sin\theta \cos^2\theta &= \frac{2}{3} \\
\int_0^\pi d\theta \sin^2\theta \cos^2\theta &= \frac{\pi}{8} \\
\int_0^\pi d\theta \sin^3\theta \cos^2\theta &= \frac{4}{15} \\
\int_0^\pi d\theta \sin^4\theta \cos^2\theta &= \frac{\pi}{16}
\end{aligned}
$$