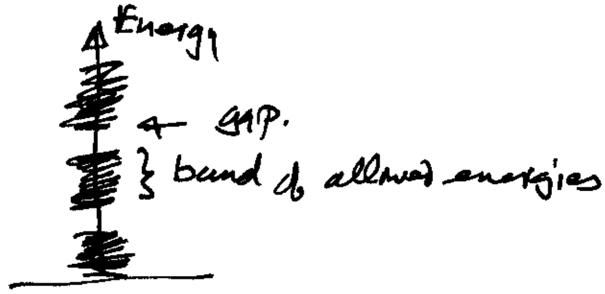
8.06 - SPRING 2005
MIDTERM SOLUTION SET

1 a) A periodic potential, no matter how weak, introduces band ~~gaps~~ gaps:



[3 points]

If n is such that all bands are completely full or completely empty, the ~~metal~~ material is an insulator. [2 points]

b) Smaller [1 point]

2. Need units of $\mathcal{E}$.

$$[\mathcal{E}] = \left[\frac{\text{Energy}}{\text{charge} \times \text{length}}\right]$$

$$[\text{charge}] = (\hbar c)^{1/2} \quad \text{— natural units}$$

$$\therefore [\alpha] = \left[\frac{\text{Energy}}{\mathcal{E}^2}\right] = \left[\frac{\text{charge}^2 \times \text{length}^2}{\text{Energy}}\right]$$

$$= \hbar c \left[\frac{L^2}{E}\right]$$

but, $[E] = \left[\frac{\hbar c}{L}\right]$

$$\therefore [\alpha] = [L^3]$$

The only length scale in the hydrogen atom problem is a_B, which by dimensional analysis is ~~$a_B \sim \frac{\hbar c}{mc^2}$~~

$$a_B \sim \frac{\hbar c}{mc^2} \times \frac{\hbar c}{e^2} = \frac{\hbar^2}{e^2 m}$$

$$\therefore \alpha = K\left(\frac{\hbar^2}{e^2 m}\right)^3$$ with K an undetermined constant.

[7 points for correct answer whether arrived at as above or not. 6 points for correctly deriving $[\alpha] = [L^3]$, and $L \sim a_B$, but getting a_B wrong.]

3. a)

$$E_{TOT} = 2\int_0^{k_F} 4\pi k^2 dk \left(\frac{L}{2\pi}\right)^3 \frac{\hbar^2 k^2}{2m} \qquad V = L^3$$

$$= \frac{\hbar^2 V}{2\pi^2 m} \frac{k_F^5}{5} \qquad [4 \text{ points}]$$

$$N = 2\int_0^{k_F} 4\pi k^2 dk \left(\frac{L}{2\pi}\right)^3$$

$$= \frac{V}{\pi^2} \frac{k_F^3}{3}$$

$$\therefore k_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \qquad [3 \text{ points}]$$

$$\therefore E_{TOT} = \frac{\hbar^2}{2\pi^2 m} \frac{(3\pi^2 N)^{5/3}}{5} V^{-2/3}$$

$$\therefore P = \frac{\hbar^2}{10\pi^2 m} (3\pi^2 N)^{5/3} \left(\frac{2}{3}\right) V^{-5/3}$$

$$= \frac{\hbar^2}{10\pi^2 m} (3\pi^2)^{5/3} \left(\frac{2}{3}\right) n^{5/3} \qquad [4 \text{ points}]$$

b) $E_{TOT} = P = 0$. [1 point]

4 a)

$$\cancel{H} \quad \vec{A} = (0, Bx, 0)$$

$$H = \frac{P_x^2}{2m} + \frac{\left(P_y - \frac{q}{c}Bx\right)^2}{2m} \qquad [3 \text{ points}]$$

b) $[H, P_y] = 0$. ∴ label eigenstates by P_y eigenvalue, $\hbar k_y$.

for such states,

$$H = \frac{P_x^2}{2m} + \frac{\left(\hbar k_y - \frac{qB}{c}x\right)^2}{2m}$$

$$= \frac{P_x^2}{2m} + \frac{1}{2}m\left(\frac{qB}{mc}\right)^2\left(x - \frac{\hbar c k_y}{qB}\right)^2$$

This is a harmonic oscillator, with $\omega \equiv \frac{qB}{mc}$.

$\therefore \quad E_n = \frac{1}{2}\hbar\omega + n\hbar\omega$, with $\omega =$ ↑ and n an integer.

[6 points]

To get the degeneracy:

The origin of the h.o. is shifted to $x_0 = \frac{\hbar c k_y}{qB}$.

In order for this to be in the box,

$$-\frac{L}{2} < x_0 < \frac{L}{2} \Rightarrow -\frac{LqB}{2\hbar c} < k_y < \frac{LqB}{2\hbar c}$$

With periodic b.c., spacing between allowed k_y is $2\pi/L$. ∴ # of allowed states is:

$$\frac{LqB}{\hbar c} \times \frac{L}{2\pi} = \frac{L^2 qB}{hc}. \qquad [6 \text{ points}]$$

4c) $\Phi = -\mathcal{E} x$

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 \left(x - \frac{\hbar c k_y}{qB}\right)^2 - q\mathcal{E} x$$

$$= \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 \left(x - \frac{\hbar c k_y}{qB} - \frac{q\mathcal{E}}{m\omega^2}\right)^2$$

$$- \frac{1}{2} m\omega^2 \left(\frac{q\mathcal{E}}{m\omega^2}\right)^2 - \frac{1}{2} m\omega^2 \, 2 \frac{\hbar c k_y q\mathcal{E}}{qB\, m\omega^2}$$

= (shifted h.o. hamiltonian)

$$- \frac{1}{2} \frac{q^2\mathcal{E}^2}{m\omega^2} - \frac{\hbar c k_y \mathcal{E}}{B}$$

$\therefore$ Energy eigenvalues $= \hbar\omega(n+\frac{1}{2}) - \frac{1}{2}\frac{q^2\mathcal{E}^2}{m\omega^2} - \frac{\hbar c k_y \mathcal{E}}{B}$

where k_y lies in the range $\frac{q\mathcal{E}}{m\omega^2} - \frac{LqB}{2\hbar c} < k_y < \frac{LqB}{2\hbar c} - \frac{q\mathcal{E}}{m\omega^2}$

[5 points. No points lost for not stating allowed range of k_y.]

5. $$\mathcal{E}_{\substack{\text{around the}\\ \text{annulus}}} = \frac{1}{2\pi r}\left(-\frac{1}{c}\right)\frac{d\Phi}{dt} \qquad \text{(faraday)}$$

$$\therefore\ j_{\substack{\text{across the}\\ \text{annulus}}} = -\frac{1}{2\pi r c}\,\frac{d\Phi}{dt}\,\frac{2e^2}{h} \qquad \left(\text{since } \sigma_H = \frac{2e^2}{h}\right)$$

[3 points]

$I \equiv$ Total current flowing radially

$$= 2\pi r \times j_{\text{across the annulus}}$$

$$= -\frac{2e^2}{hc}\,\frac{d\Phi}{dt}$$

[2 points]

Number of electrons passing across the ammeter between $t=0$ and $t=T$

$$= \frac{1}{e}\int_0^T dt\, I = -\frac{2e}{hc}\int_0^T dt\,\frac{d\Phi}{dt}$$

$$= -\frac{2e}{hc}\left[\Phi(T) - \Phi(0)\right]$$

$$= -2\,\frac{\Phi(T) - \Phi(0)}{\Phi_0}$$

[3 points]

where $\Phi_0 \equiv \frac{hc}{e}$.

[Note: No points lost for ignoring the overall sign. You were anyway not told which direction the "ammeter" is connected.]

5 cont.

The Aharanov Bohm effect analysis tells us that if $\Phi(T) - \Phi(0)$ is an integer multiple of Φ_0, then the wave function at T is identical to that at 0. We need this in order to conclude that the current through the ammeter is the same as that we calculated. [2 points]

6 a) $E = \dfrac{\hbar^2 \ell(\ell+1)}{2M}$ degeneracy $= (2\ell+1)$

$\therefore\ \ell = 0 :\quad E = 0 \quad$ degen $= 1$

$\ell = 1 :\quad E = \dfrac{\hbar^2}{M} \quad$ degen $= 3$ [2 points]

b) $H' = -\dfrac{q}{2Mc} B_0 L_z$

H' already ~~degenerate~~ diagonal in standard $|\ell, m\rangle$ basis.

$\therefore$ Energies of $\ell = 1$ states are:

$$\underbrace{\frac{\hbar^2}{M}}_{E^0} + \underbrace{\left(-\frac{qB_0}{2Mc}\right)}_{E^1} \hbar \underbrace{m}_{m = 0, \pm 1}$$

[4 points]

c) $H' = \epsilon xy = \epsilon \underbrace{r^2}_{=1} \sin^2\theta \sin\phi \cos\phi$

$$H' = \epsilon \sin^2\theta \sin\phi \cos\phi = \frac{\epsilon}{2} \sin^2\theta \sin 2\phi = \frac{\epsilon}{4i} \overbrace{\ }^{\sin^2\theta} \left(e^{2i\phi} - e^{-2i\phi}\right)$$

Must evaluate $\langle 1\, m_a | H' | 1\, m_b \rangle$ for $m_a = 1, 0, -1$
$m_b = 1, 0, -1$.

$$\langle 1\, m_a | H' | 1\, m_b \rangle = \frac{\epsilon}{4i} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\ Y^*_{1 m_a} \sin^2\theta \left(e^{2i\phi} - e^{-2i\phi}\right) Y_{1 m_b}$$

Since $Y_{1m} \sim e^{im\phi} f(\theta)$, the ϕ integral vanishes unless $m_b - m_a = \pm 2$.

6 c) cont.

So, the nonzero elements are $\langle 1 1 | H' | 1 -1 \rangle$

and $\langle 1 -1 | H' | 1 1 \rangle$

which are complex conjugates

Evaluating, we find

$$\langle 1 -1 | H' | 1 1 \rangle = \frac{\epsilon}{4i} \underbrace{(2\pi)}_{\phi \text{ integral}} \int_0^\pi \sin\theta \, d\theta \left(+ \sqrt{\frac{3}{8\pi}} \sin\theta \right) \sin^2\theta \left(- \sqrt{\frac{3}{8\pi}} \sin\theta \right)$$

$$= - \frac{\epsilon}{4i} (2\pi) \frac{3}{8\pi} \underbrace{\int_0^\pi \sin^5\theta \, d\theta}_{16/15}$$

$$= + \frac{i\epsilon}{5}$$

So, in the $|1 m\rangle$ basis, H' is:

$$\begin{array}{ccc} |1-1\rangle & |10\rangle & |11\rangle \end{array}$$

$$\begin{pmatrix} 0 & 0 & i\frac{\epsilon}{5} \\ 0 & 0 & 0 \\ -i\frac{\epsilon}{5} & 0 & 0 \end{pmatrix}$$

which has eigenvalues $0, \pm \frac{\epsilon}{5}$

$\therefore$ To 1st order, the energies are

$$\frac{\hbar^2}{M} + \frac{\epsilon}{5}, \quad \frac{\hbar^2}{M}, \quad \frac{\hbar^2}{M} - \frac{\epsilon}{5} .$$

[11 points]

6 d) $H' = -q\mathcal{E}_0 z = -q\mathcal{E}_0 \underset{=1}{\underset{\uparrow}{r}} \cos\theta = -q\,\mathcal{E}_0 \cos\theta$

$\therefore\ H' = -q\mathcal{E}_0 \sqrt{\frac{4\pi}{3}}\, Y_{10}$

$l=0$ state not degenerate.

1st order correction $\sim \langle 00|\cos\theta|00\rangle = 0$ [2 points]

2nd order correction involves sum over matrix elements

$\langle 00|\, H' | lm\rangle$ for all $|lm\rangle$ except $|00\rangle$.

$$\langle 00|H'|lm\rangle = -q\mathcal{E}_0 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, \frac{1}{\sqrt{4\pi}} \sqrt{\frac{4\pi}{3}}\, Y_{10}^* Y_{lm}$$

Because the Y_{lm} are orthonormal, this vanishes for all l,m except $|l,m\rangle = |10\rangle$.

Only nonzero element is:

$$\langle 00|\, H'|10\rangle = -q\mathcal{E}_0\, (2\pi)\, \frac{1}{\sqrt{3}} \int_0^{\pi} \sin\theta\, d\theta\, Y_{10}^2$$

$$= -q\mathcal{E}_0\, \frac{2\pi}{\sqrt{3}} \int_0^{\pi} \sin\theta\, d\theta\, \frac{3}{4\pi} \cos^2\theta$$

$$= -q\mathcal{E}_0\, \frac{2\pi}{\sqrt{3}}\, \frac{3}{4\pi}\, \frac{2}{3}$$

$$= -\frac{q\mathcal{E}_0}{\sqrt{3}}$$

6d) cont:

$\therefore$ 2nd order shift

$$= \sum_{lm} \frac{|\langle 00 | H' | lm \rangle|^2}{E^0_{00} - E^0_{lm}}$$

$$= \frac{|-q\mathcal{E}_0/\sqrt{3}|^2}{0 - \hbar^2/M}$$

$$= -\frac{q^2 \mathcal{E}_0^2 M}{3\hbar^2}$$

$\therefore$ gs energy to 2nd order is

$$\underset{E^0_{00}}{\underset{\uparrow}{0}} + \underset{E^1_{00}}{\underset{\uparrow}{0}} - \underset{E^2_{00}}{\underset{\uparrow}{\frac{q^2 \mathcal{E}_0^2 M}{3\hbar^2}}}$$

[6 points]