Quantum Physics III (8.06) Spring 2006
Midterm Solution

Mar 23, 2006

1. **Short Answer (16 points)**

   (a) (1 point) B.

   (b) (2 points) (i) False (ii) True

   (c) (3 points) $\frac{e^2}{h}$. One can deduce it by dimensional analysis as follows. The Hall conductivity σ_H has the dimension of velocity. On dimensional ground it must have the form

   $$\sigma_H \sim \alpha^n c, \qquad \alpha = \frac{e^2}{\hbar c}$$

   for some number n, since α is the only dimensionless constant one can form out of fundamental constant $\hbar, c, e$. Since σ_H arises in non-relativistic quantum mechanics, it should not depend on c, leading to $n = 1$.

   (d) Turning on a weak periodic potential introduces band gaps in the single-particle energy spectrum (3 points). If the number density n is such that all energy bands are completely full or empty, the material is an insulator (2 points).

   (e) One could use a double slit experiment with a solenoid hidden behind the screen to measure the Aharonov-Bohm effect (2 points). In classical mechanics electromagnetic potentials are convenient mathematical devises to compute the fields (1 point). In quantum mechanics they have independent physical meaning as demonstrated by the Aharonov-Bohm effect (2 points).

2. **Dimensional Analysis of a Degenerate Fermi Gas (10 points)**

   (a) (5 points) For a *non-relativistic* Fermi gas, the energy of a single particle is proportional to $1/m$, so is the energy density ϵ. Thus $K \propto \frac{1}{m}$ (1 point). In natural units, with $\hbar = c = 1$, K then has dimension $1/eV$. Also note that the volume has dimension $(eV)^{-3}$ in natural units. Thus the energy density ϵ has dimension $(eV)^4$ and the number density n has dimension $(eV)^3$ and equation $\epsilon = Kn^\gamma$ becomes

   $$(eV)^4 = \frac{1}{eV}(eV)^{3\gamma}$$

   We find that $\gamma = \frac{5}{3}$.

   (b) (5 points) For an *ultra-relativistic* Fermi gas, the energy of a single particle is independent of m, so are ϵ and K (1 point). K is thus dimensionless in natural units. Since energy density ϵ has unit eV^4 and number density n has dimension eV^3, we find that $\gamma = \frac{4}{3}$.

3. **Fermi surface (10 points)**

(a) (3 points) To find the ground state energy, we fill electrons to the lowest available single-particle levels allowed by the Pauli exclusion principle. Since an electron has two spin states, we find that

$$E_0 = 2\sum_{n=0}^{3}(n+\frac{1}{2})\hbar\omega = 16\hbar\omega$$

(b) (3 points) For the first excited state, we excite one electron at level $n = 3$ to level $n = 4$, so we have

$$E_1 = E_0 + \hbar\omega = 17\hbar\omega$$

(c) For 8 bosons, we simply fill the lowest single particle state for ground state (2 points)

$$E_0 = 8 \times \frac{1}{2}\hbar\omega = 4\hbar\omega$$

and the first excited state has energy (2 points)

$$E_1 = E_0 + \hbar\omega = 5\hbar\omega$$

4. **Perturbations of a Two-Dimensional Harmonic Oscillator (26 points)**

(a) The ground state of H_0 is $|00\rangle$. We have

$$H'|00\rangle = \Delta|1,1\rangle \tag{1}$$

Thus the first order correction is (2 points)

$$E_{00}^{(1)} = \langle 0,0|H'|0,0\rangle = 0$$

and the second order correction is (4 points)

$$E_{00}^{(2)} = \sum_{n,m\neq 0} \frac{|\langle 0,0|H'|n,m\rangle|^2}{E_{00}^{(0)} - E_{nm}^{(0)}} = \frac{|\langle 0,0|H'|1,1\rangle|^2}{E_{00}^{(0)} - E_{11}^{(0)}} = -\frac{\Delta^2}{2\hbar\omega}$$

where only $n = m = 1$ term contributes in the sum due to (1).

(b) The unperturbed states of energy $2\hbar\omega$ are $|0,1\rangle$ and $|1,0\rangle$ which are two-fold degenerate (2 points). One finds that

$$\langle 0,1|H'|0,1\rangle = \langle 1,0|H'|1,0\rangle = 0$$

and

$$\langle 0,1|H'|1,0\rangle = \langle 1,0|H'|0,1\rangle = \Delta$$

Thus in this degenerate subspace we have (4 points)

$$H' = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

with eigenvalues (2 points)

$$E_{\pm}^{(1)} = \pm\Delta$$

(c) The eigenvector for $E_+^{(1)} = \Delta$ is (2 points)

$$\psi_+^{(0)} = \frac{1}{\sqrt{2}}\left(|0,1\rangle + |1,0\rangle\right)$$

and the eigenvector for $E_-^{(1)} = -\Delta$ is (2 points)

$$\psi_+^{(0)} = \frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right)$$

(d) Note that

$$\langle 1,2|H'|0,1\rangle = \sqrt{2}\Delta, \qquad \langle 2,1|H'|1,0\rangle = \sqrt{2}\Delta$$

which leads to

$$\langle 1,2|H'|\psi_+\rangle = \Delta, \qquad \langle 2,1|H'|\psi_+\rangle = \Delta$$

These are the only nonzero matrix elements outside the degenerate subspace to which ψ_+ belongs. Thus one finds that the second order correction to the energy of ψ_+ is

$$E_+^{(2)} = \frac{|\langle 1,2|H'|\psi_+\rangle|^2}{E_{01}^{(0)} - E_{12}^{(0)}} + \frac{|\langle 2,1|H'|\psi_+\rangle|^2}{E_{01}^{(0)} - E_{21}^{(0)}} = -\frac{\Delta^2}{\hbar\omega}$$

The second order energy correction to ψ_+ is exactly the same, i.e.

$$E_-^{(2)} = -\frac{\Delta^2}{\hbar\omega}$$

(e) We expect the perturbative expansion to be good if

$$\Delta \ll \hbar\omega$$

5. **Particle in a magnetic field and harmonic oscillator potential (15 points)**

(a) Since $V(x) = \frac{1}{2}m\omega_0^2 x^2$ depends explicitly on x, we will choose a gauge which depends on x only,

$$\vec{A} = (0, Bx, 0)$$

and the time-independent Schrodinger equation in this gauge is given by

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}\left(p_y - \frac{q}{c}Bx\right)^2 + \frac{1}{2}m\omega_0^2 x^2$$

(b) In the absence of the potential $V(x)$, the Hamiltonian is

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}\left(p_y - \frac{q}{c}Bx\right)^2$$

Since

$$[p_y, H] = 0$$

we can write the wave function as

$$\psi(x, y) = e^{ik_y y} f(x)$$

Acting on such a wave function the Hamiltonian becomes

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2} m\omega_L^2 (x - x_0)^2$$

with

$$\omega_L = \frac{qB}{mc}, \qquad x_0 = \frac{c\hbar k_y}{qB}$$

Thus the energy spectrum is given by

$$E_n = (n + \frac{1}{2})\hbar\omega_L, \qquad n = 0, 1, \cdots$$

(c) In the presence of the potential $V(x)$, we still have

$$[p_y, H] = 0$$

The wave function can then be written as

$$\psi(x, y) = e^{ik_y y} f(x)$$

Acting on such a wave function the Hamiltonian becomes

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2} m\omega_L^2 (x - x_0)^2 + \frac{1}{2} m\omega_0^2 x^2$$

with

$$\omega_L = \frac{qB}{mc}, \qquad x_0 = \frac{c\hbar k_x}{qB}$$

One can further write H as (using $\omega_0 = \omega_L$)

$$H = \frac{1}{2m} p_x^2 + \frac{1}{2}(2m\omega_L^2)(x - x_0/2)^2 + \frac{m\omega_L^2}{4} x_0^2$$

This is a harmonic oscillator of frequency $\sqrt{2}\omega_L$ with a constant energy shift. Thus we find

$$E_n(k_y) = (n + \frac{1}{2})\sqrt{2}\hbar\omega_L + \frac{m\omega_L^2}{4} x_0^2 = (n + \frac{1}{2})\sqrt{2}\hbar\omega_L + \frac{\hbar^2 k_x^2}{4m}, \qquad n = 0, 1, \cdots$$