

## SUPPLEMENTARY NOTES ON CANONICAL QUANTIZATION AND APPLICATION TO A CHARGED PARTICLE IN A MAGNETIC FIELD

These notes were prepared by Prof. Jaffe for the 1998 version of Quantum Physics III.

Classical mechanics is internally consistent. No amount of examination of Newton's Laws as an abstract system will lead you to quantum mechanics. The quantum world forced itself upon us when physicists tried and failed to explain the results of experiments using the tools of classical mechanics. It took and still takes considerable guesswork to find the proper description of a new quantum system when first encountered. Notions like internal spin and the Pauli exclusion principle have no analog whatsoever in classical physics.

However, the equations of motion of quantum mechanics, looked at from a particular point of view, resemble the Hamiltonian formulation of classical mechanics. This similarity has led to a program for *guessing* the quantum description of systems with classical Hamiltonian formulations. The program is known as "canonical quantization" because it makes use of the "canonical" *i.e.* Hamiltonian, form of classical mechanics. Though it is very useful and quite powerful, it is important to remember that it provides only the first guess at the quantum formulation. The only way to figure out the complete quantum mechanical description of a system is through experiment. Also, recall from 8.05 that there are many quantum mechanical systems (like the spin-1/2 particle, for example) whose Hamiltonians cannot be obtained by canonically quantizing some classical Hamiltonian.

We will begin 8.06 by applying the method of canonical quantization to describe the motion of a charged particle in a constant magnetic field. In so doing, we shall discover several beautiful, and essentially quantum mechanical, phenomena: Landau levels, the integer quantum Hall effect, and the Aharonov-Bohm effect. Our treatment will be self-contained, and thus of necessity will barely scratch the surface of the subject. We will, however, be able to grasp the essence of several key ideas and phenomena. This part of 8.06 serves as an introduction to the condensed matter physics of electrons in materials at low temperatures in high magnetic fields, which is

a vast area of contemporary experimental and theoretical physics. The integer and fractional quantum hall effect (we shall not treat the fractional case) were both discovered in experiments done the 1980's, and were among the biggest surprises in physics of recent decades. They have been the subject of intense investigation ever since, including by many physicists here at MIT. Furthermore, understanding gauge invariance and phenomena like the Aharonov-Bohm effect are key aspects of the modern understanding of the theories that govern the interactions of all known elementary particles.

And now, Prof. Jaffe's notes...

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# 1 Canonical Quantization

## 1.1 The canonical method

There is a haunting similarity between the equations of motion for operators in the Heisenberg picture and the *classical* Hamilton equations of motion in *Poisson bracket* form.

First let's summarize the quantum equations of motion. Consider a system with  $N$  degrees of freedom. These could be the coordinates of  $N/3$  particles in three dimensions or of  $N$  particles in one dimension for example. Generically we label the coordinates  $\{x_j\}$  and the momenta  $\{p_j\}$ , where  $j = 1, 2, \dots, N$ . We denote the Hamiltonian  $\mathcal{H}(x, p)$ , where we drop the subscripts on the  $x$ 's and  $p$ 's if no confusion results. From wave mechanics where  $p_j$  is represented by  $-i\hbar\partial/\partial x_j$ ,

$$\begin{aligned} [x_j, x_k] &= 0 \\ [p_j, p_k] &= 0 \\ [x_j, p_k] &= i\hbar\delta_{jk} \end{aligned} \tag{1.1}$$

From Ehrenfest's equation (for a general operator),  $\mathcal{A}_H(t)$ , in the Heisenberg picture,

$$i\hbar \frac{\partial \mathcal{A}}{\partial t} = [\mathcal{A}, \mathcal{H}] \tag{1.2}$$

(where we suppress the subscript  $H$  on the operator) we can obtain equations of motion for the  $\{x_j\}$  and  $\{p_j\}$ ,

$$\begin{aligned} i\hbar\dot{x}_j &= [x_j, \mathcal{H}] \\ i\hbar\dot{p}_j &= [p_j, \mathcal{H}]. \end{aligned} \quad (1.3)$$

In the particular case where  $\mathcal{H} = \sum_{j=1,N} (p_j^2/2m) + V(x)$ , it is easy to extend the work we did in the case of the harmonic oscillator to obtain from (1.3)

$$\begin{aligned} \dot{p}_j &= -\frac{\partial \mathcal{H}}{\partial x_j} = -\frac{\partial V}{\partial x_j} \\ \dot{x}_j &= \frac{\partial \mathcal{H}}{\partial p_j} = \frac{p_j}{m} \end{aligned} \quad (1.4)$$

Equations (1.4) look exactly like Hamilton's equations. When the two lines are combined, we obtain Newton's second law,  $m\ddot{x}_j = -\partial V/\partial x_j$ . Of course, we have to remember that the *content* of these equations is very different in quantum mechanics than in classical mechanics: operator *matrix elements* between states are the observables, and the states cannot have sharp values of both  $x$  and  $p$ . Nevertheless (1.4) are identical in *form* to Hamilton's equations and the similarity has useful consequences.

In fact, it is the form of (1.2) and (1.3) that most usefully connects to classical mechanics. Let's now turn to the *Poisson Bracket* formulation of Hamilton's equations for classical mechanics. We have a set of  $N$  *canonical coordinates*  $\{x_j\}$  and their *conjugate momenta*  $\{p_j\}$ . Suppose  $A$  and  $B$  are any two dynamical variables — that is, they are characteristics of the system depending on the  $x$ 's and the  $p$ 's. Examples of dynamical variables include the angular momentum,  $\vec{L} = \vec{x} \times \vec{p}$ , or the kinetic energy,  $\sum_j p_j^2/2m$ . Then the *Poisson Bracket* of  $A$  and  $B$  is defined by,

$$\{A, B\}_{\text{PB}} \equiv \sum_{j=1}^N \left\{ \frac{\partial A}{\partial x_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial x_j} \right\}. \quad (1.5)$$

Poisson Brackets are introduced into classical mechanics because of the remarkably simple form that Hamilton's equations take when expressed in terms of them,

$$\dot{x}_j = \frac{\partial H}{\partial p_j} = \{x_j, H\}_{\text{PB}}$$

$$\dot{p}_j = -\frac{\partial H}{\partial x_j} = \{p_j, H\}_{\text{PB}} \quad (1.6)$$

as can easily be verified by using the definition of the PB on the dynamical variables  $x_j$ ,  $p_j$ , and  $H$ . The time development of an arbitrary dynamical variable can also be written simply in terms of Poisson Brackets. For simplicity we consider dynamical variables that do not depend explicitly on the time.\* Then

$$\begin{aligned} \dot{A} \equiv \frac{dA}{dt} &= \sum_{j=1}^N \left\{ \frac{\partial A}{\partial x_j} \dot{x}_j + \frac{\partial A}{\partial p_j} \dot{p}_j \right\} \\ &= \{A, H\}_{\text{PB}} \end{aligned} \quad (1.7)$$

where the second line follows from the first by substituting from (1.6) for  $\dot{x}$  and  $\dot{p}$ .

Finally, to complete the analogy, note that the Poisson Brackets of the  $x$ 's and the  $p$ 's themselves are remarkably simple,

$$\begin{aligned} \{x_j, x_k\} &= 0 \\ \{p_j, p_k\} &= 0 \\ \{x_j, p_k\} &= \delta_{jk} \end{aligned} \quad (1.8)$$

because  $\partial x_j / \partial p_k = 0$ ,  $\partial x_j / \partial x_k = \delta_{jk}$ , etc.

Now we can step back and compare the Poisson Bracket formulation of classical mechanics with the operator equations of motion of quantum mechanics. Compare (1.1) to (1.8), (1.2) to (1.7) and (1.3) to (1.6). It appears that a classical Hamiltonian theory can be transcribed into quantum mechanics by the simple rule,

$$\{A, B\}_{\text{PB}} \Rightarrow \frac{1}{i\hbar} [\mathcal{A}, \mathcal{B}]. \quad (1.9)$$

where the quantum operators  $\mathcal{A}$  and  $\mathcal{B}$  are the same functions of the operators  $\hat{x}_j$  and  $\hat{p}_j$  as  $A$  and  $B$  are of  $x_j$  and  $p_j$ .

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\*A dynamical variable may or may not depend *explicitly* upon the time. Any dynamical variable will depend *implicitly* on the time through the variables  $x_j$  and  $p_j$ . Explicit time dependence arises when some agent external to the system varies explicitly with the time. An example is the time dependence of the magnetic interaction energy,  $-\vec{\mu} \cdot \vec{B}(t)$ , when an external magnetic field depends on time.

This remarkable rule tells us how to *guess* the quantum theory corresponding to a given classical dynamical system. The procedure is called “canonical quantization” because it follows from the canonical Hamiltonian description of the classical dynamics. In fact there are some important limitations of the canonical quantization method that will be discussed in a later subsection. First, however, let’s summarize (and appreciate the elegance of) the simple steps necessary to find the quantum equivalent of a classical Hamiltonian system —

- Set up the classical Hamiltonian dynamics in terms of canonical coordinates  $\{x_j\}$  and momenta  $\{p_j\}$ , with a Hamiltonian  $H$ .
- Write the equations of motion in Poisson Bracket form.
- Reinterpret the classical dynamical variables as quantum operators in a Hilbert space of states. The commutation properties of the quantum operators are determined by the rule (1.9).

Of course we cannot forget the difference between quantum and classical mechanics: Although the fundamental equations of motion can be placed in correspondence by the canonical quantization procedure, the different interpretation of classical and quantum variables leads to totally different pictures of phenomena.

## 1.2 Simple Examples

Here are some simple examples of the canonical quantization procedure. Later we will encounter a very important and non-trivial example in the problem of a charged particle moving in a magnetic field.

### 1.2.1 Bead on a Wire

Suppose a rigid wire is laid out in space along a curve  $\vec{X}(s)$ . We parameterize the wire by a single coordinate  $s$  which measures length along the wire. Let a bead slide without friction along the wire. This is a standard (easy) problem in Lagrangian mechanics. The energy of the bead is entirely kinetic and is given by  $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{s}^2$ , because the bead is restricted to move only along the curve. So the Lagrangian is  $L = \frac{1}{2}m\dot{s}^2$ ; the momentum conjugate to  $s$  is

$p = \partial L / \partial \dot{s} = m\dot{s}$ ; the Hamiltonian is  $H = p^2/2m$ ; and the quantum theory is defined by the operators  $\hat{s}$ ,  $\hat{p}$  and  $\mathcal{H} = \hat{p}^2/2m$ . In short the bead behaves like a free particle on a line. It experiences no forces due to the curving of the wire.

We can dress up this problem a little by adding gravity. Suppose the wire is placed in a constant gravitational field  $\vec{g} = -g\hat{y}$ . Now there is a potential energy  $V(s) = mgy(s)$ . The canonical operators are still  $\hat{s}$  and  $\hat{p}$ , but now the Hamiltonian is  $\mathcal{H} = \hat{p}^2/2m + mgy(\hat{s})$ .

This is actually so oversimplified a problem that interesting physics has been lost. A real bead is held on a real wire by some force that prevents it moving in the directions transverse to the wire. A simple model of this would be to replace the wire by a *tube* whose center follows the track of the wire, but has a finite (say constant circular) cross section. Then the particle would be free to move in the tube, but unable to leave it. This is a more complicated problem, but it can be analyzed pretty simply in the limit that the transverse dimension of the tube,  $d$ , is small compared to the radius of curvature,  $R(s)$ , of the wire.<sup>† ‡</sup> The end result is the appearance of a new term in  $\mathcal{H}$  proportional to the inverse of the squared radius of curvature,

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + mgy(\hat{s}) - \frac{1}{4R(\hat{s})^2}. \quad (1.10)$$

So a real bead on a wire feels a force that *attracts* it to the regions where the wire is most curved ( $R(s)$  is smallest).

### 1.2.2 Relative and Center of Mass Coordinates

Consider two particles moving in three dimensions and interacting with one another by a force that depends only on their relative separation. The classical Lagrangian is

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 - V(\vec{r}_1 - \vec{r}_2). \quad (1.11)$$

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<sup>†</sup>The radius of curvature at some point  $\vec{X}(s)$  is the radius of a circular disk that is adjusted to best approximate the wire at the point  $\vec{X}$ .

<sup>‡</sup>This problem has attracted attention recently in connection with the propagation of electrons in “quantum wires”. If you’re interested, Prof. Jaffe can supply you with references, since he wrote many of them.

We could quantize this canonically in this form and obtain a two particle Schroedinger equation. Instead let's make the transformation to relative and center of mass coordinates at the classical level and quantize from there. We define

$$\begin{aligned}\vec{r} &= \vec{r}_1 - \vec{r}_2 \\ \vec{R} &= \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2)\end{aligned}\tag{1.12}$$

where  $M = m_1 + m_2$  is the total mass of the system. Substituting into (1.11) we find

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}).\tag{1.13}$$

where  $\mu$  is the *reduced mass* of the two body system. Now is it easy to read off  $\vec{P}$ , the momentum canonically conjugate to  $\vec{R}$ ,  $\vec{P} = m\dot{\vec{R}}$ , and  $\vec{p}$ , the momentum canonically conjugate to  $\vec{r}$ ,  $\vec{p} = \mu\dot{\vec{r}}$ . The transition to quantum mechanics is accomplished by postulating the canonical commutators,  $[\hat{R}_j, \hat{P}_k] = i\hbar\delta_{jk}$ ,  $[\hat{r}_j, \hat{p}_k] = i\hbar\delta_{jk}$ , etc. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2M}\hat{\vec{P}}^2 + \frac{1}{2\mu}\hat{\vec{p}}^2 + V(\hat{\vec{r}}).\tag{1.14}$$

So just like classical mechanics, the center of mass moves like a free particle, and the interesting part of the dynamics is just like a single particle of mass  $\mu$  moving in a potential  $V(\vec{r})$ .

These examples may seem overly elementary. If this were all canonical quantization was good for, it would not be necessary for us to spend much time on it. Moreover there are many mistakes to be made by applying the canonical method too naively (as we shall see below). In fact, canonical quantization helps us guess the quantum equivalent of some highly non-trivial classical systems like charged particles moving in electromagnetic fields, and the dynamics of the electromagnetic field itself.

### 1.3 Warnings

The canonical quantization method is not a *derivation* of quantum mechanics from classical mechanics. The substitution (1.9) cannot be motivated within

classical mechanics. It represents a guess, or a leap of the imagination, forced on us by the bizarre phenomena that were observed by the early atomic physicists and that were inexplicable within the confines of classical mechanics. The canonical quantization method is simply a recognition that the quantum mechanics of a single particle that was developed from wave mechanics is in fact a representative of a class of systems — those described by traditional Hamiltonian mechanics — that all can be quantized by the same methods.

Many systems we are interested in quantizing differ from this norm. The attitude I would like to advocate is that we use canonical quantization as the first step toward a quantum equivalent of a classical theory, but that we remain open minded about the need to augment or refine the quantum theory if phenomena force us. Some of the problems that arise on the road from classical to quantum mechanics are listed below in order of increasing severity (in my opinion).

### 1.3.1 Quantum variables without classical analogues

There are some systems that possesses quantum degrees of freedom that — for one reason or another — do not persist in the classical regime. The best known example is *spin*. When the quantum states of electrons were studied in the 1920's, it was soon discovered that the electron possesses other degrees of freedom that classical point particles don't have. Using guesswork and experimental information, physicists invented operators that describe the behavior of this innately quantum mechanical variable. Nature has forced us to postulate new quantum variables to augment the classical description of a system. This does not represent a failure of quantum mechanics. Quite the opposite — one of its great strengths is that phenomena *without classical analog* can be introduced relatively easily, without upsetting the basic framework of the theory.

Many great advances of 20th century physics (spin, color, internal symmetries, *etc.*) fall into the category of discovering and understanding quantum variables without classical analog. Physicists relish the possibility of such radical departures from classical dynamics.

### 1.3.2 Operator ordering ambiguities

Classical dynamical variables commute with one another, so the order in which they are written does not affect the dynamics. Not so in quantum mechanics. Suppose, our Lagrangian was  $\frac{m}{2} \frac{\xi^2 \dot{\xi}^2}{R^2}$ .<sup>§</sup> Then the classical Hamiltonian would have been  $H = \frac{1}{2m} \frac{p^2}{\xi^2}$ . When  $p$  and  $\xi$  become operators, how is this to be interpreted? Is it  $\frac{1}{\xi} p^2 \frac{1}{\xi}$ ? or  $p \frac{1}{\xi^2} p$ ? or some other variant. Using  $[\xi, p] = i\hbar$  it is easy to see that the different variants yield different Schroedinger wave equations and therefore different physics.

This problem is called an “operator ordering ambiguity”. More physics input is required to eliminate the ambiguity. Sometimes general principles help: a Hamiltonian must be hermitian in order that probability be conserved. If the ambiguous term  $xp$  (note  $xp \neq px$ ) occurred in a Hamiltonian, we could rather confidently replace it by the hermitian form  $\frac{1}{2}(xp + px)$ . Sometimes, hermiticity is not enough. General conservation laws, like conservation of momentum or angular momentum help. If all else fails, it is necessary to leave the ambiguity (parameterized by the relative strength of different hermitian combinations) and see which best describes experiment. In practice I am not aware of physically important examples where hermiticity and conservation laws fail to resolve operator ordering ambiguities.

### 1.3.3 Singular points

The canonical quantization method becomes complicated and subtle when one tries to apply it to coordinate systems that include singular points. A familiar example is spherical polar coordinates ( $r$ ,  $\theta$ , and  $\phi$ ). The origin,  $r = 0$ , is a singular point for spherical polar coordinates — for example,  $\theta$  and  $\phi$  are not defined at  $r = 0$ . If you follow the canonical formalism through from Lagrangian to canonical momenta ( $p_r$ ,  $p_\theta$ , and  $p_\phi$ ), to Hamiltonian, to canonical commutators, a host of difficulties arise. Although it is possible to sort them out by insisting that the canonical momenta be hermitian operators, it is considerably easier to quantize the system in Cartesian coordinates and make the change to spherical polar coordinates at the quantum level. This is the path taken in most elementary treatments of quantum mechanics in three dimensions: the operator  $\vec{p}^2 = p_1^2 + p_2^2 + p_3^2$  is recognized as the

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<sup>§</sup>This is the lagrangian of a point mass on a string wrapping around a spool of radius  $R$ . The coordinate  $\xi$  is the length of string (assumed straight) not wound up.

Laplacian in coordinate representation ( $p_j \leftrightarrow -i\hbar\partial/x_j \Rightarrow \vec{p}^2 \leftrightarrow -\hbar^2\vec{\nabla}^2$ ) and the transformation to polar coordinates is made by writing the Laplacian and the wavefunction in terms of  $r$ ,  $\theta$ , and  $\phi$ . As a rule of thumb: the canonical approach becomes cumbersome when the classical coordinates and/or momenta do not range over the full interval from  $-\infty$  to  $+\infty$ .

### 1.3.4 Constrained systems

Finally we must at least mention a complicated and rich variation of the canonical quantization method that has become an important focus for research in recent years. Sometimes the degrees of freedom of complicated systems are not all independent. For example, a particle may be constrained to move on a specified surface (in three dimensions). Then the coordinates and velocities that appear in the Lagrangian cannot be regarded as independent variables. The changes in  $x$ ,  $y$  and  $z$  must be correlated so that the particle remains on the surface. The canonical formalism can break down in several (related ways). Sometimes one (or more) of the canonical momenta is identically zero. If  $p_k \equiv 0$ , then the associated Hamilton's equation,  $\partial H/\partial q_k = 0$  is not an equation of motion. Instead it is a constraint that must be satisfied by the canonical coordinates and momenta at each time. The constraint may not be consistent with canonical commutation relations. A simple, but not particularly interesting example, would be the constraint  $x+y=0$  imposed on motion in two dimensions. The constraint is not consistent with the canonical commutator  $[x, p_y] = 0$  and  $[y, p_y] = i\hbar$  because the commutators can be added to give  $[x+y, p_y] = i\hbar$ . This case is not serious because we could return to the original lagrangian, use the constraint to eliminate one dynamical variable from the problem, and then proceed without difficulty. In this case, we would write  $\xi = x - y$  and  $\eta = x + y$  and use the constraint to eliminate  $\eta$  from the problem. In more complicated cases it is not possible to remove the constraint in this fashion, either because it is too hard to solve the constraint equations for one or more variables, or because the problem has some deep underlying symmetry that would be broken by choosing to solve for and eliminate one variable as opposed to another. Dirac realized the importance of such problems and developed a method to handle quantization under constraints. Other powerful yet practical methods were developed by L. D. Faddeev and V. N. Popov in the 1960's. Quantum versions of electrodynamics, chromodynamics (the theory of the interactions of

quarks and gluons) and gravity all make use of these modern extensions of the idea of canonical quantization.

## 2 Motion in a Constant Magnetic Field – “Landau Levels”

### 2.1 Introduction

A pretty and relatively simple application of canonical methods in quantum theory is to the motion of a charged particle in a constant magnetic field. This problem was first solved by the great Russian theoretical physicist, Lev Landau. In recent years condensed matter physicists have found interesting applications of Landau’s problem to real physical systems.

Suppose a magnetic field  $\vec{B}_0$ , constant in magnitude, direction, and time fills a region of space. For definiteness we assume  $\vec{B}_0$  points in the  $\hat{e}_3$  direction. All points in the  $xy$ -plane are equivalent — a simple example of *translation invariance*. The classical motion of a charged particle in a constant magnetic field is determined by the Lorentz force law,

$$m\ddot{\vec{x}} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B}. \quad (2.15)$$

When  $\vec{E} = 0$  the force always acts at right angles to the velocity, so kinetic energy is conserved. For simplicity, we restrict the motion to the  $xy$ -plane. Then the particle moves in a circle at a constant angular velocity,

$$\begin{aligned} \frac{mv^2}{r} &= \frac{e}{c}vB_0 \\ \omega_L &\equiv \frac{v}{r} = \frac{eB_0}{mc} \end{aligned} \quad (2.16)$$

$\omega_L$  is known as the Larmor frequency. (You can also find it called the cyclotron frequency.)

Quantizing this system requires all the apparatus of the canonical quantization method we developed earlier in these notes. To analyze the problem quantum mechanically we must first find the classical Hamiltonian that describes the system. Then we will be able to go over to the quantum domain.

This requires us to introduce the concept of a *vector potential*,  $\vec{A}$ , which determines the magnetic field by its *curl*,

$$\vec{B} = \vec{\nabla} \times \vec{A} . \quad (2.17)$$

Once we have the Hamiltonian, we will solve the quantum problem, illustrate some of the subtleties, and apply what we have learned to the *Aharonov-Bohm* and *Integer Quantum Hall Effects*. A recurring theme in much of this study is how the system reflects the underlying translation invariance in the  $xy$ -plane. The magnetic field is independent of  $x$  and  $y$ . The Hamiltonian does not respect this symmetry, however, because the vector potential depends on  $x$  and  $y$  in an asymmetric manner. In the end the physics must be translation invariant but it will take some work to demonstrate this.

## 2.2 Classical Hamiltonian

This section and the next rewritten by KR to avoid mention of the Lagrangian. Since this year many 8.06 students are not very familiar with Lagrangians and Hamiltonians in classical mechanics, I will never mention the Lagrangian. It is after all only the Hamiltonian that is needed for our quantum mechanical purposes.

We now need the classical Hamiltonian which describes the dynamics of a charged particle in a constant magnetic and electric field, moving according to the Lorentz force law (2.15). (Let's allow for a nonzero electric field, in addition to the magnetic field. We'll need this later.) A detailed *derivation* of the Hamiltonian is more appropriately the subject of a course in electrodynamics. Here I will tell you the answer and show that the Hamilton equations reduce to the Lorentz force law. The Hamiltonian is:

$$\mathcal{H} = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + e\phi , \quad (2.18)$$

where the electrostatic potential  $\phi$  and the vector potential  $\vec{A}$  describe the magnetic and electric fields via (2.17) and

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} . \quad (2.19)$$

Note that the Hamiltonian  $\mathcal{H}$  depends on  $\vec{x}$  because  $\vec{A}$  and  $\phi$  do. We must now check that Hamilton's equations reproduce the Lorentz force law.

The first Hamilton equation is just

$$\dot{x}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i - \frac{e}{c} A_i}{m} \quad (2.20)$$

or

$$m\dot{x}_i = p_i - \frac{e}{c} A_i . \quad (2.21)$$

Note that the canonical momentum  $\vec{p}$  is NOT the same as the kinematical momentum  $m\vec{v}$ . You should not be disturbed about this. Your intuition about what constitutes the canonical momentum conjugate to the coordinate  $x$  relies upon momentum conservation, and momentum is not conserved in this problem. So that you do not feel completely at sea, note that

$$\mathcal{H} = \frac{1}{2} m \vec{v}^2 + e\phi \quad (2.22)$$

which is a sensible expression for the energy of the particle: the magnetic field acts at right angles to the particle's velocity and therefore does not contribute to its energy. Starting with (2.22), however, and guessing that  $\vec{p} = m\vec{v}$  is *wrong*: this would lead to a  $\mathcal{H}(\vec{x}, \vec{p})$  whose Hamilton's equations fail to reproduce the Lorentz force law. As we shall now confirm, the correct choice is (2.18), meaning that  $\vec{p}$  and  $m\vec{v}$  are related by (2.21)

The second Hamilton equation is:

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial x_i} = \frac{e}{mc} \sum_{k=1}^3 \left( p_k - \frac{e}{c} A_k \right) \frac{\partial A_k}{\partial x_i} - e \frac{\partial \phi}{\partial x_i} . \quad (2.23)$$

Henceforth, we shall not write the  $\sum$  symbol, understanding that any repeated indices (like  $k$  above) are summed from 1 to 3. We now substitute the first Hamilton equation into the second and obtain

$$\dot{p}_i = \frac{e}{c} \dot{x}_k \frac{\partial A_k}{\partial x_i} - e \frac{\partial \phi}{\partial x_i} , \quad (2.24)$$

where we have employed the implied summation notation.

We now differentiate (2.21) with respect to time to obtain another expression for  $\dot{p}_i$ :

$$\dot{p}_i = m\ddot{x}_i + \frac{e}{c} \dot{A}_i + \frac{e}{c} \frac{\partial A_i}{\partial x_k} \dot{x}_k . \quad (2.25)$$

The last term here is a “convective term”: the  $\vec{A}$  experienced by the particle can change with time either because  $\vec{A}$  changes in time or because the particle moves and  $\vec{A}$  changes in space. The convective term is obtained by the chain rule, upon noting that we are always interested in  $\vec{A}(\vec{x}(t))$ . Note that we have used the implied summation notation in writing this term.

We now combine (2.24) and (2.25) and use (2.19) to obtain:

$$m\ddot{x}_i = \frac{e}{c} \dot{x}_k \left( \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right) + eE_i . \quad (2.26)$$

It is now just a matter of applying various identities to see that this is in fact the Lorentz force law. Noting that

$$\frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} = (\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}) \frac{\partial A_n}{\partial x_m} = \epsilon_{ikj} \epsilon_{jmn} \frac{\partial A_n}{\partial x_m} , \quad (2.27)$$

wherein we have introduced our old friends the Kronecker  $\delta$  the antisymmetric  $\epsilon$  and made repeated use of the implied summation notation, we find

$$m\ddot{x}_i = \frac{e}{c} \epsilon_{ikj} \dot{x}_k \epsilon_{jmn} \frac{\partial A_n}{\partial x_m} + eE_i . \quad (2.28)$$

Now, we just have to remember that the curl of any two vectors  $\vec{U}$  and  $\vec{V}$  is defined as  $(\vec{U} \times \vec{V})_i = \epsilon_{ijk} U_j V_k$  to see that we have derived the Lorentz force law

$$m\ddot{\vec{x}} = \frac{e}{c} \dot{\vec{x}} \times \vec{\nabla} \times \vec{A} + e\vec{E} = \frac{e}{c} \vec{v} \times \vec{B} + e\vec{E} \quad (2.29)$$

from the Hamiltonian  $\mathcal{H}$  of (2.18). Voilà.

## 2.3 Canonical Quantization

To quantize, we postulate canonical commutation relations among operators  $\vec{x}$  and  $\vec{p}$ :

$$\begin{aligned} [x_j, p_k] &= i\hbar \delta_{jk} \\ [x_j, x_k] &= [p_j, p_k] = 0. \end{aligned} \quad (2.30)$$

and interpret the Hamiltonian  $\mathcal{H}(\vec{x}, \vec{p})$  of (2.18) as the quantum mechanical Hamiltonian operator. It is crucial that we have correctly identified the

classical canonical momenta  $\vec{p}$ , such that the Hamilton equations obtained from  $\mathcal{H}(\vec{x}, \vec{p})$  describe the physics. It is only once we have done so that we know how to apply the canonical quantization procedure. That is, it is only once we have done so that we know what  $\vec{p}$  operators satisfy (2.30), and know the quantum mechanical Hamiltonian operator.

In order to write the Hamiltonian explicitly, we must prescribe a vector potential corresponding to  $\vec{B}_0 = B_0 \hat{e}_3$ . We choose

$$\vec{A} = \frac{B_0}{2} (x_1 \hat{e}_2 - x_2 \hat{e}_1). \quad (2.31)$$

It is easy to verify that  $\vec{\nabla} \times \vec{A} = B_0 \hat{e}_3$ , however other choices such as  $\vec{A} = B_0 x_1 \hat{e}_2$  or  $\vec{A} = -B_0 x_2 \hat{e}_1$  would do just as well. They all describe the same magnetic field. In classical mechanics we know that physics depends only on  $\vec{B}$ . The same is true here in the Landau problem, even though at this moment we have only the Hamiltonian which does depend on the choice of  $\vec{A}$ . We shall verify in lecture that although different choices of vector potential do result in different Hamiltonians, they yield the same energy eigenvalues.

Substituting the explicit choice (2.31) for  $\vec{A}$  into the Hamiltonian  $\mathcal{H}$  of (2.18), we find

$$\begin{aligned} \mathcal{H} &= \frac{1}{2m} \left\{ \left( p_1 + \frac{eB_0}{2c} x_2 \right)^2 + \left( p_2 - \frac{eB_0}{2c} x_1 \right)^2 \right\} + \frac{1}{2m} p_3^2 \\ &= \frac{1}{2m} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} m \omega^2 (x_1^2 + x_2^2) - \omega L_3 \end{aligned} \quad (2.32)$$

where  $\omega = \frac{1}{2} \omega_L = eB_0/2mc$ , and  $L_3 = x_1 p_2 - x_2 p_1$  is the angular momentum in the  $x_1 - x_2$  plane. So the system looks like a particle in a two-dimensional harmonic oscillator with an additional potential  $-\omega L_3$ . Having solved the harmonic oscillator before, we can easily construct the energy eigenvalues and eigenstates for this problem.

## 2.4 A solution to the quantum equations of motion

First consider conservation laws: Apparently angular momentum about the  $\hat{e}_3$  - axis is conserved,  $[L_3, \mathcal{H}] = 0$ . So is momentum in the  $\hat{e}_3$  direction,  $[p_3, \mathcal{H}] = 0$ . First, we dispose of the  $p_3$  dependence by restricting the problem to motion in the  $x_1 - x_2$  plane. Less formally we could equally well diagonalize

$p_3$ , labeled our eigenstates by its eigenvalue,  $k_3$ , and add  $(1/2m)k_3^2$  to our eigenenergies.

The other components of momentum, however, are not conserved:

$$[p_j, \mathcal{H}] \neq 0 \quad (2.33)$$

for  $j = 1, 2$ . This comes as a surprise: since the magnetic field is uniform in space we would expect the system to be translation invariant, and therefore to find momentum conserved. On the other hand, the classical motion is circular, so perhaps we should not be surprised that the usual concept of momentum has to be amended. Resurrecting momentum conservation will be a principal task in the following analysis.

To classify the eigenstates of (2.32) we introduce standard harmonic oscillator “creation” and “annihilation” operators (in the following we replace the coordinates  $x$  and  $y$  by  $x_1$  and  $x_2$ ),

$$x_k \equiv \sqrt{\frac{\hbar}{2m\omega}}(a_k + a_k^\dagger) \quad p_k \equiv -i\sqrt{\frac{\hbar m\omega}{2}}(a_k - a_k^\dagger), \quad (2.34)$$

and  $a_\pm \equiv \frac{1}{\sqrt{2}}(a_1 \mp ia_2)$ , with the usual commutation relations,

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \\ [a_j, a_k] &= [a_\pm, a_\pm] = [a_\mp, a_\pm] = 0 \\ [a_\pm, a_\pm^\dagger] &= 1, \end{aligned} \quad (2.35)$$

Substituting into  $\mathcal{H}$  and  $L_3$  we obtain

$$\begin{aligned} L_3 &= \hbar(a_+^\dagger a_+ - a_-^\dagger a_-), \\ \mathcal{H} &= \hbar\omega(a_+^\dagger a_+ + a_-^\dagger a_- + 1) - \hbar\omega(a_+^\dagger a_+ - a_-^\dagger a_-) \\ &= \hbar\omega_L\left(a_-^\dagger a_- + \frac{1}{2}\right), \end{aligned} \quad (2.36)$$

which has a straightforward, though unusual interpretation. Eigenstates of  $\mathcal{H}$  and  $L_3$  are labeled by the number of quanta of  $+$  and  $-$  excitation,

$$\begin{aligned} \mathcal{N}_\pm &\equiv a_\pm^\dagger a_\pm \\ \mathcal{N}_\pm |n_+, n_-\rangle &= n_\pm |n_+, n_-\rangle \end{aligned} \quad (2.37)$$

$\pm$  quanta carry  $\pm 1$  unit of angular momentum, but *only*  $n_-$ - quanta contribute to the energy. So the energy eigenstates, known as *Landau levels*, are each infinitely degenerate. After exploring some of the properties of Landau levels we will return and try to sort out this degeneracy.

To summarize:

- There are two physical conserved quantities, the energy  $E$  and  $z$ -component of angular momentum,  $L$ .
- Equivalently, the number of “oscillator quanta”,  $n_+$  and  $n_-$  are conserved, with
  - $E(n_+, n_-) = \hbar\omega_L(n_- + \frac{1}{2})$ , and
  - $L(n_+, n_-) = \hbar(n_+ - n_-)$ .
- $E$  is independent of  $n_+$ , and  $n_-$  can take on any non-negative integer value, so each Landau energy level is infinitely degenerate.
- For energy  $(n_- + \frac{1}{2})\hbar\omega_L$  the *tower* of degenerate states begins at angular momentum  $-n_-$  and grows in steps of  $\hbar$  to infinity.

## 2.5 Physical Interpretation of Landau Levels

The results of the last section are as puzzling as they are enlightening. Classically a particle moving in the  $x_1$ - $x_2$  plane under the influence of a constant magnetic field,  $B_0\hat{e}_3$ , can have any energy and its (circular) orbit can be centered at any point  $(x_1, y_1)$ . In the quantum world the states are gathered into discrete energy levels separated by  $\hbar\omega_L$ . Each level is vastly degenerate. The Landau Hamiltonian, eq. (2.32), on the other hand, singles out a specific origin of coordinates about which the harmonic oscillator potential is centered. In this section we explore the meaning of the Landau degeneracy and (eventually) explain how the translational invariance so obvious at the classical level, is manifest in the quantum theory.

### 2.5.1 The location and size of Landau levels

Despite superficial appearances each energy eigenstate found in the previous section can be placed wherever we wish in the  $x_1$ - $x_1$  plane. The simplest

way to see this uses the coherent state formalism that was developed in 8.05. Since the energy depends only on  $n_-$  we can superpose states with different  $n_+$  without leaving a fixed energy level. Define, then, the coherent state

$$|\alpha\rangle \equiv \exp \alpha a_+^\dagger |0, 0\rangle \quad (2.38)$$

where  $|0, 0\rangle$  is the eigenstate of eq. (2.32) with  $n_+ = n_- = 0$ , and  $\alpha$  is an arbitrary complex number. Eq. (2.38) creates a coherent state in the  $+$  oscillator but leaves  $n_- = 0$  untouched. Using eq. (2.34) and eq. (2.35) it is easy to show that

$$\begin{aligned} \langle x_1 \rangle &= \ell_0 \operatorname{Re} \alpha \\ \langle x_2 \rangle &= \ell_0 \operatorname{Im} \alpha, \end{aligned} \quad (2.39)$$

where  $\ell_0 = \sqrt{\hbar/m\omega}$ , so by choosing the real and imaginary parts of  $\alpha = (x_1 - ix_2)/\ell_0$  we can center a state with energy  $E_0 = \frac{1}{2}\hbar\omega_L$  wherever we wish.

Momentum and velocity are not directly proportional in this problem. The state  $|\alpha\rangle$  provides a graphic example: The quantum equation of motion for  $x_k$  tells us,

$$\dot{x}_k = \frac{1}{i\hbar} [x_k, H] \quad (2.40)$$

so it is easy to show that the expectation value of the velocity is zero in the state  $|\alpha\rangle$ :

$$\begin{aligned} \langle \alpha | \dot{x}_k | \alpha \rangle &= \frac{1}{i\hbar} \langle \alpha | [x_k, H] | \alpha \rangle \\ &= \frac{1}{i\hbar} \langle \alpha | x_k E - E x_k | \alpha \rangle = 0. \end{aligned} \quad (2.41)$$

so the state centered at  $(x_1, x_2)$  does not wander away with time. On the other hand the components of the momenta  $p_1$ , and  $p_2$  do not vanish in the state  $|\alpha\rangle$ ,

$$\begin{aligned} \langle p_1 \rangle &= \frac{\hbar}{\ell_0} \operatorname{Im} \alpha \\ \langle p_2 \rangle &= \frac{\hbar}{\ell_0} \operatorname{Re} \alpha, \end{aligned} \quad (2.42)$$

but this has no direct physical interpretation because the momenta not simply  $mv$ .<sup>¶</sup>

In contrast to its position, the size of a Landau level is directly connected to its energy and to the strength of the external magnetic field. To see this we make a semiclassical estimate of the area,  $\mathcal{A}(n_-)$ . Since  $v/r = \omega_L$ , and classically  $E = \frac{1}{2}mv^2$ , we have  $r^2 = E/2m\omega_L^2$ . Now the quantum theory requires (2.36), so  $r^2 = (\hbar/m\omega_L)(n_- + \frac{1}{2})$ . This becomes more transparent if we multiply by  $\pi B_0$  to form the *flux* through the orbit,  $\Phi_L(n_-) \equiv \pi r^2 B_0$ , and substitute for  $\omega_L$ ,

$$\Phi_L(n_-) = \frac{hc}{e}(n_- + \frac{1}{2}) \quad (2.43)$$

It appears that the flux through the particle's orbit comes in units of a *fundamental quantum unit of flux*,  $\Phi_0 \equiv hc/e$ ! In the next section, we shall see that something very close to this characterizes the full quantum treatment.

### 2.5.2 A more careful look at translation invariance

In this section we take a more sophisticated approach to the Landau problem that will clarify both the degeneracy of the Landau levels and the way in which the system manages to respect homogeneity in the  $xy$ -plane. We begin with a canonical transformation, trading  $x_j$  and  $p_j$  for new variables,

$$\begin{aligned} \Pi &= p_1 + m\omega x_2 \\ \phi &= \frac{1}{2}\left(x_1 - \frac{p_2}{m\omega}\right) \\ P_1 &= p_1 - m\omega x_2 \\ P_2 &= p_2 + m\omega x_1 \end{aligned} \quad (2.44)$$

It is easy to check that the Landau Hamiltonian, (2.18), can be written in terms of  $\Pi$  and  $\phi$  alone,

$$\mathcal{H} = \frac{1}{2m}\Pi^2 + \frac{m}{2}\omega_L^2\phi^2, \quad (2.45)$$

---

<sup>¶</sup>We shall see in lecture that  $\langle p_k \rangle$  is not gauge invariant, and therefore cannot correspond to a physical observable.  $\langle mv_k \rangle = \langle p_k - (e/c)A_k \rangle$  is gauge invariant. KR.

with  $[\phi, \Pi] = i\hbar$ . Once again, we have a harmonic oscillator, with eigenenergies  $E_n = (n + \frac{1}{2})\hbar\omega_L$ .

The degeneracy formerly associated with  $n_-$  is now connected with the  $P_1$  and  $P_2$ . Since  $[P_k, \phi] = [P_k, \Pi] = 0$  we see that  $P_1$  and  $P_2$  are candidate constants of the motion. To see what symmetry they generate consider the commutators of  $P_k$  with  $x_j$ , the spatial coordinates. A brief calculation yields

$$[x_j, P_k] = i\hbar\delta_{jk} \quad (2.46)$$

So the new “momenta”,  $P_k$ , generate translations of the coordinates in the usual sense. Thus the symmetry associated with the constants of the motion (and the degeneracy of the Landau levels) is translation invariance. This is a welcome result. However, we cannot simply diagonalize  $P_1$  and  $P_2$  along with  $\mathcal{H}$  because *they do not commute with each other*,

$$[P_1, P_2] = -2im\omega\hbar. \quad (2.47)$$

Since the commutator of  $P_1$  and  $P_2$  is a  $c$ -number, perhaps we can construct *functions* of them which do commute. The simplest possibility is to look at finite translations of the form,

$$\begin{aligned} \mathcal{T}_1(b_1) &\equiv e^{-i\frac{P_1 b_1}{\hbar}} \\ \mathcal{T}_2(b_2) &\equiv e^{-i\frac{P_2 b_2}{\hbar}}. \end{aligned} \quad (2.48)$$

According to our study of translations, these operators should translate  $x_1$  and  $x_2$  by  $b_1$  and  $b_2$  respectively. Indeed, it is easy to show that on account of (2.46),

$$\begin{aligned} \mathcal{T}_1(b_1)^\dagger x_1 \mathcal{T}_1(b_1) &= x_1 + b_1 \\ \mathcal{T}_1(b_1)^\dagger x_2 \mathcal{T}_1(b_1) &= x_2 \end{aligned} \quad (2.49)$$

and so on.

The crucial question is whether we can choose  $b_1$  and  $b_2$  such that the finite translations commute,  $[\mathcal{T}_1(b_1), \mathcal{T}_2(b_2)] = 0$ . First, let's determine the

effect of  $\mathcal{T}_1(b_1)$  on  $P_2$ ,

$$\begin{aligned}
f_2(b_1) &\equiv \mathcal{T}_1(b_1)^\dagger P_2 \mathcal{T}_1(b_1) \\
\frac{df_2}{db_1} &= \frac{-i}{\hbar} \mathcal{T}_1(b_1)^\dagger [P_1, P_2] \mathcal{T}_1(b_1) \\
&= -2m\omega, \quad \text{so} \\
f_2(b_1) &= f_2(0) - 2m\omega b_1 \\
&= P_2 - 2m\omega b_1.
\end{aligned} \tag{2.50}$$

This enables us to apply the operator  $\mathcal{T}_1(b_1)$  to  $\mathcal{T}_2(b_2)$ ,

$$\begin{aligned}
\mathcal{T}_1^\dagger(b_1) \mathcal{T}_2(b_2) \mathcal{T}_1(b_1) &= e^{-2ib_1 b_2 m\omega/\hbar} \mathcal{T}_2(b_2), \quad \text{or} \\
\mathcal{T}_2(b_2) \mathcal{T}_1(b_1) &= e^{-2ib_1 b_2 m\omega/\hbar} \mathcal{T}_1(b_1) \mathcal{T}_2(b_2).
\end{aligned} \tag{2.51}$$

So the finite translations fail to commute only by virtue of this multiplicative factor of unit magnitude. If (and only if) we choose the parameters  $b_1$  and  $b_2$  so that the phase is a multiple of  $2\pi$  then the translations commute. This condition is

$$2m\omega b_1 b_2 / \hbar = 2\pi N \tag{2.52}$$

for an integer  $N$ . This defines a rectangle of area  $b_1 b_2$  in the  $xy$ -plane. Let us find the flux through this rectangle,

$$\Phi(b_1, b_2) = b_1 b_2 B_0 = \frac{\pi \hbar}{m\omega} N = N \Phi_0 \tag{2.53}$$

So we have established a maximal set of commuting operators for the Landau problem:  $\mathcal{H}$ ,  $\mathcal{T}_1(b_1)$ , and  $\mathcal{T}_2(b_2)$ , where  $b_1$  and  $b_2$  obey (2.53) completely characterize the states of a charged particle in a constant magnetic field.

Let's take some time to interpret this result...

**Translations in the  $n_+, n_-$  basis** Now that we know that  $\mathcal{T}_1(b_1)$  and  $\mathcal{T}_2(b_2)$ , commute with  $\mathcal{H}$  when  $\vec{b} \equiv (b_1, b_2)$  satisfies (2.53), we can use them to translate the states we found in §2.4 around the  $xy$ -plane. The motivation for this is to understand the degeneracy of the Landau levels associated with

the  $n_-$  quantum number. Since the state with  $n_+ = n_- = 0$  is localized at the origin, we anticipate that the translation by  $\vec{b}$  will produce a state localized around  $\vec{b}$ .

Consider, then, the state with  $n_+ = n_- = 0$  — a harmonic oscillator ground state centered at  $\vec{x} = 0$ . Now translate this state to  $\vec{b}$ , where  $(b_1, b_2)$  satisfies the condition (2.53) with  $N = 1$ , so they represent the smallest translation that commutes with the Hamiltonian,

$$|\vec{b}, 0, 0\rangle \equiv \mathcal{T}_1(b_1)\mathcal{T}_2(b_2)|0, 0\rangle. \quad (2.54)$$

This state is normalized to unity because the operators  $\mathcal{T}$  are unitary. It has energy  $E = \frac{1}{2}\hbar\omega_L$  and  $\langle\vec{x}\rangle = \vec{b}$ . What does it look like in terms of the original basis  $|n_+, n_-\rangle$ ? To answer this we must express the translations in terms of the  $\{a_k\}$ . Using the Baker-Hausdorf Theorem, a short calculation gives:

$$|\vec{b}, 0, 0\rangle = \exp\left(-\frac{1}{2}\frac{|b|^2}{\ell_0^2}\right) \exp\left(\frac{b^*a_+^\dagger}{\ell_0}\right)|0, 0\rangle, \quad (2.55)$$

where  $b \equiv b_1 - ib_2$  and  $\ell_0 = \sqrt{\hbar/m\omega}$  is the natural scale of lengths associated with the Landau problem. This is an example of a “coherent state”, as we discussed in §2, superposing an infinite series of degenerate states with differing values of  $n_+$ . Note that  $n_- = 0$  is preserved so, as promised, translation did not change the energy of the state. The prefactor is of particular interest because it determines the overlap of the translated state with the original state,  $|0, 0\rangle$ ,

$$|\langle 0, 0|\vec{b}, 0, 0\rangle|^2 = \exp\left(-2\pi\frac{b_1^2 + b_2^2}{2b_1b_2}\right). \quad (2.56)$$

The exponential is bounded by  $e^{-2\pi}$ . Thus the overlap of the original state and its translation to the next “cell” is very small. The same analysis applies to any of the eigenstates  $|0, n_-\rangle$ . We can translate each of these energy eigenstates to any point on rectangular lattice  $(n_1b_1, n_2b_2)$  throughout the plane. Although these states are not orthonormal, they they give us a qualitatively correct picture of the solutions of the Landau problem as towers of nearly localized energy eigenstates with  $E = (n + \frac{1}{2})\hbar\omega_L$  situated in unit cells on a grid labeled by any pair of distances  $b_1$  and  $b_2$  satisfying (2.53).

**Eigenstates of the finite translations** Another, more conventional, approach is to study eigenstates of our maximal set of commuting operators,  $\mathcal{H}$ ,  $\mathcal{T}_1(b_1)$ , and  $\mathcal{T}_2(b_2)$ . First we choose some  $b$  satisfying (2.53). Since  $\mathcal{T}_k$  is a unitary operator its eigenvalues are complex numbers of unit magnitude, which we parameterize by

$$\begin{aligned}\mathcal{T}_1(b_1)|\phi_1, \phi_2, n\rangle &= e^{i\phi_1}|\phi_1, \phi_2, n\rangle \\ \mathcal{T}_2(b_2)|\phi_1, \phi_2, n\rangle &= e^{i\phi_2}|\phi_1, \phi_2, n\rangle \\ \mathcal{H}|\phi_1, \phi_2, n\rangle &= (n + \frac{1}{2})\hbar\omega_L|\phi_1, \phi_2, n\rangle\end{aligned}\tag{2.57}$$

It is easy to see that the phase  $\phi_j$  must be linear in  $b_j$  ( $\mathcal{T}_1(2b_1) = [\mathcal{T}_1(b_1)]^2$ ), so we define  $\phi_j \equiv k_j b_j$ , where  $k_j$  is a real number in range  $-\pi/b_j < k_j < \pi/b_j$  because the phase  $\phi$  is only defined modulo  $2\pi$ . This makes these states look very similar to plane waves even though they are not. If we construct the coordinate space wavefunction corresponding to  $|\phi_1, \phi_2, n\rangle$ ,

$$\psi_{k_1 k_2 n}(x_1, x_2) \equiv \langle x_1, x_2 | \phi_1, \phi_2, n \rangle \tag{2.58}$$

then it is easy to see that the consequence of (2.57) is that

$$\psi_{k_1 k_2 n}(x_1 + b_1, x_2 + b_2) = e^{ik_1 b_1 + ik_2 b_2} \psi_{k_1 k_2 n}(x_1, x_2) \tag{2.59}$$

just like a plane wave,  $\exp i(k_1 x_1 + k_2 x_2)$ , would behave. The difference, of course is that  $\psi$  has this simple behavior only for the special translations we have discovered, not for an arbitrary translation, and as a consequence, the “momenta”  $(k_1, k_2)$  are not conserved.

States like these arise in many other situations where a system is invariant only under certain *finite* translations. The classic example is a crystal lattice which is invariant if we translate by the vectors that define a *unit cell* but not otherwise. In this way the Landau system resembles a two-dimensional crystal with a rectangular unit cell whose area is determined by the flux quantization condition (2.53). Wavefunctions that behave like (2.59) are known as Bloch waves in honor of Felix Bloch who first studied quantum mechanics in periodic structures. We will discuss them in detail when addressing the quantum theory of electrons in metals.

Note that the states  $\psi_{k_1 k_2 n}(x_1, x_2)$  extend everywhere throughout the  $xy$ -plane. This is clear from (2.59) since their amplitude arbitrary distances away

from some original location is only modulated by a phase. Thus this basis is very different from the quasi-localized basis we obtained by translating the state  $|n_+ = 0, n_- = 0\rangle$  around the plane in the previous section. Of course they describe the same system and the same physics and are related through the marvelous power of the superposition principle.

### 3 The Aharonov Bohm Effect

The vector potential,  $\vec{A}(\vec{x})$  makes a surprising appearance in the quantum description of a particle in a magnetic field. It all stems from the classical Hamiltonian,

$$\mathcal{H} = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2. \quad (3.60)$$

Despite the appearance of  $\vec{A}$  in  $\mathcal{H}$ , we know that at the *classical* level, the dynamics depends only on  $\vec{E}$  and  $\vec{B}$  because only they appear in Newton's Laws, eq. (2.15). In the classical domain  $\vec{A}$  and the electrostatic potential,  $\phi$  can be regarded as merely useful, but inessential, abstractions.

In the quantum theory  $\mathcal{H}$  rather than  $m\ddot{x}$  is fundamental, so the possibility exists that *physics* depends on  $\vec{A}$ . For the case we have studied in detail — motion in a constant magnetic field —  $\vec{A} = \frac{1}{2}\vec{x} \times \vec{B}$  so we cannot even define dependence on  $\vec{A}$  independent of  $\vec{B}$ .

In 1959 Y. Aharonov and D. Bohm proposed a way to observe a direct effect of  $\vec{A}$  and established the quantum significance of  $\vec{A}$ .<sup>||</sup> Although this may seem like a somewhat technical detail, it captures some of the unusual aspects of “reality” in the quantum world and has fascinated students ever since. Perhaps more important, vector potentials associated with generalizations of electromagnetism play a central role in our extraordinarily successful theories of subnuclear particle physics. In that arena observable consequences of the vector potentials abound.

Aharonov and Bohm proposed to consider motion of a charged particle in the plane perpendicular to an idealized solenoid that produces a constant magnetic field,  $\vec{B} = B_0\hat{e}_3$ , but *only within a circle of radius  $R$* . For  $r > R$ ,  $\vec{B}$  can be taken to vanish identically. This configuration is shown in Fig. 1 along with a couple of paths that will figure in the discussion. Even though

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<sup>||</sup>Y. Aharonov and D. Bohm, *Phys. Rev.* **115** (1959) 485.

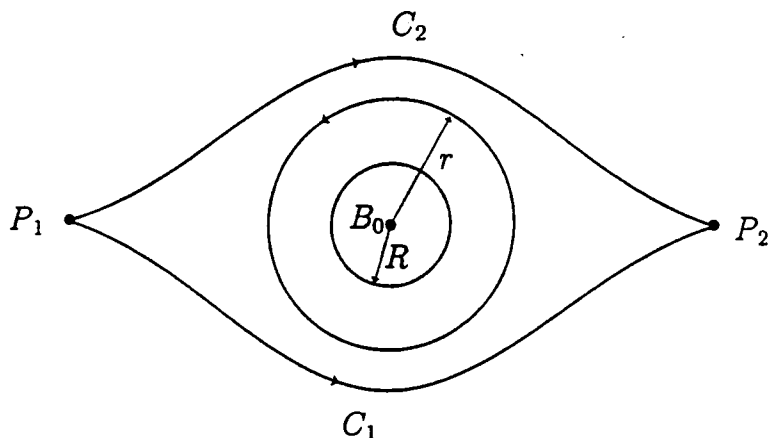


Figure 1: Paths for analysis of Aharonov-Bohm effect.

$\vec{B} = 0$  for  $r > R$ ,  $\vec{A}$  cannot vanish in this region because of Stokes' theorem. Consider the integral of the vector potential around the circle marked  $C$  (which is the boundary of a disk  $S$ ) of radius  $r$  shown in the figure. Then Stokes' theorem and the defining relation for  $\vec{A}$ ,  $\vec{B} = \vec{\nabla} \times \vec{A}$ , give

$$\begin{aligned} \oint_C \vec{dl} \cdot \vec{A} &= \iint d^2S \hat{e}_3 \cdot \vec{\nabla} \times \vec{A} \\ &= \iint d^2S \hat{e}_3 \cdot \vec{B} = \pi R^2 B_0, \end{aligned} \quad (3.61)$$

so  $\vec{A}$  cannot vanish everywhere on the circle  $C$ . In fact symmetry requires that  $\vec{A}$  point in the azimuthal,  $\hat{\phi}$ , direction, so an elementary calculation gives,

$$\vec{A} = \frac{\Phi}{2\pi r} \hat{\phi} \quad \text{for } r > R \quad (3.62)$$

Of course the existence of a vector potential in the region outside  $R$ , where  $\vec{B} = 0$ , is a classical phenomenon no more surprising than the existence of an electrostatic potential inside a uniformly charged sphere where  $\vec{E} = 0$ . The question of interest is whether some physical phenomenon that takes place entirely in the region  $r > R$ , where  $\vec{B} = 0$  can depend on  $\vec{A}$ . To study this, consider a charged particle described by a wave packet that moves on either of the two paths marked  $C_1$  and  $C_2$  in the figure. Of course a quantum particle cannot be constrained to a definite path, but we will see that the

effect is the same along all paths close to  $C_1$  or  $C_2$ , so a diffuse path of the sort allowed in quantum mechanics will give the same result.

The particle's propagation through the vector potential is determined by the time dependent Schrödinger equation,

$$\frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2\psi(\vec{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r}, t), \quad (3.63)$$

where  $\vec{p} = -i\hbar\vec{\nabla}$ . Define the Aharonov-Bohm phase factor,

$$g(\vec{r}, C) = \frac{e}{\hbar c} \int_C^{\vec{r}} \vec{dl} \cdot \vec{A} \quad (3.64)$$

where the line integral begins at the point  $P_1$ , follows curve  $C$  and ends at a point  $\vec{r}$ . Note that

$$\vec{\nabla}g(\vec{r}, C) = \frac{e}{\hbar c}\vec{A}(\vec{r}) \quad (3.65)$$

independent of the curve  $C$ . Now factor the phase  $g$  out of the wavefunction,

$$\psi(\vec{r}, t) = \exp(ig) \chi(\vec{r}, t), \quad (3.66)$$

and substitute into eq. (3.63). The result is that  $\chi$  obeys the free Schrödinger equation,

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\chi = i\hbar\dot{\chi}. \quad (3.67)$$

Thus all information about the vector potential is contained in the phase that multiplies  $\chi$ .

With the from of eq. (3.66) in mind let us compare the phase accumulated by a well-localized charged particle wave packet that begins at point  $P_1$  and propagates to  $P_2$  along *either* path  $C_1$  or  $C_2$ . We start with  $\psi(\vec{r}, t_1)$  concentrated at  $P_1$  at  $t = t_1$ . Let us suppose, however, that  $\psi$  is a quantum mechanical superposition of two terms,  $\psi(\vec{r}, t_1) = \psi_1(\vec{r}, t_1) + \psi_2(\vec{r}, t_1)$  such that  $\psi_1(\vec{r}, t_1)$  and  $\psi_2(\vec{r}, t_1)$  describe wave packets which subsequently (i.e. after  $t_1$ ) follow the two different paths  $C_1$  and  $C_2$ . At time  $t_2$ , the two wave packets both reach  $P_2$ , and

$$\psi(\vec{r}, t_2) = \exp\left[\frac{ie}{\hbar c} \int_{C_1} \vec{dl} \cdot \vec{A}\right] \chi_1(\vec{r}, t_2) + \exp\left[\frac{ie}{\hbar c} \int_{C_2} \vec{dl} \cdot \vec{A}\right] \chi_2(\vec{r}, t_2) \quad (3.68)$$

with both  $\chi_1(\vec{r}, t_2)$  and  $\chi_2(\vec{r}, t_2)$  localized in the vicinity of the point  $P_2$ . Whatever other ( $\vec{A}$ -independent) relative phase may have accumulated by

the time the wave packets have reached  $P_2$ , there is an  $\vec{A}$ -dependent *relative* phase,

$$\psi(\vec{r}, t_2) = \exp \left[ \frac{ie}{\hbar c} \int_{C_1} \vec{dl} \cdot \vec{A} \right] \left\{ \chi_1(\vec{r}, t_2) + \exp \left[ \frac{ie}{\hbar c} \oint_{\bar{C}} \vec{dl} \cdot \vec{A} \right] \chi_2(\vec{r}, t_2) \right\}. \quad (3.69)$$

Note that the relative phase is given by the loop integral over the closed path  $\bar{C} = C_2 - C_1$ .

The relative phase in eq. (3.69) is measurable, for example by watching the interference pattern on a detector at  $P_2$  as the magnetic field,  $\vec{B}$  is slowly changed. The phase depends only on the loop integral of  $\vec{A}$ , which in turn depends only on the total magnetic field enclosed within the path  $\bar{C}$ ,

$$\begin{aligned} g(\bar{C}) &\equiv \frac{e}{\hbar c} \oint_{\bar{C}} \vec{dl} \cdot \vec{A} \\ &= e\pi R^2 B_0 / \hbar c = e\Phi / \hbar c \end{aligned} \quad (3.70)$$

where  $\Phi = \pi R^2 B_0$  is the magnetic flux contained within  $\bar{C}$ .

We have been fairly careful in this discussion to make it clear that the particle moves entirely in a region where  $\vec{B} = 0$ , so the only source of the phase is the vector potential  $\vec{A}$ . The result does not depend on the details of the path followed by the particle. For example, if we replace the path  $C_2$  by a nearby path  $C'_2$  on the same side of the solenoid, then the resulting phase,  $g(\bar{C}')$ , where  $\bar{C}' = C'_2 - C_1$  is unchanged because  $g(\bar{C}')$  depends only on the magnetic flux enclosed by  $\bar{C}'$ , which is the same as that enclosed by  $\bar{C}$ . Thus it does not matter that the quantum particle cannot follow a sharp trajectory. The Aharonov-Bohm phase is a global property of the motion, not a property of the particle's exact path. A similar argument shows that  $g(\bar{C})$  does not depend on the gauge we choose to describe the vector potential. If we change gauge, from  $\vec{A}$  to  $\vec{A}'$ , with

$$\vec{A}' = \vec{A} - \vec{\nabla}\Lambda,$$

then

$$g(\bar{C}) \rightarrow g'(\bar{C}) = g(\bar{C}) - \frac{e}{\hbar c} \oint_{\bar{C}} \vec{dl} \cdot \vec{\nabla}\Lambda = g(\bar{C}) \quad (3.71)$$

because the integral of the gradient of any continuous function around a closed path is zero.

So Aharonov and Bohm have shown in this simple example, that the vector potential has physical manifestations in quantum mechanics. Although

the phase  $g(\bar{C})$  depends on the magnetic flux enclosed by the path,  $\bar{C}$ , the path itself lies entirely in a region of space in which  $\vec{B} = 0$  and  $\vec{A} \neq 0$ .

To quote Griffiths, page 349, “What are we to make of the Aharonov-Bohm effect? Evidently our classical preconceptions are simply *mistaken*. There *can* be electromagnetic effects in regions where the fields [ $\vec{B}$  and  $\vec{E}$ ] are zero. Note, however, that this does not make  $\vec{A}$  itself measurable — only the enclosed *flux* comes into the final answer, and the theory remains gauge invariant.” You should read Griffiths, section 10.2.4, but should for now ignore the connection to Berry’s phase.

The Aharonov-Bohm effect does not only manifest itself as shifts in interference patterns. See pages 344-345 in Griffiths for a description of how the Aharonov-Bohm effect leads to shifts in the *energy levels* for a “bead on a loop of string” if the string is everywhere in a region with  $\vec{B} = 0$  but encircles a flux carrying solenoid.

## 4 Integer Quantum Hall Effect

The Hall Effect is an elementary electromagnetic phenomenon where a conducting strip carrying a current along its length develops a current across its width when placed in a magnetic field. The direction of the induced current is sensitive to the *sign* of the electric charge of mobile species in the material and can be used to show that conventional currents are carried by electrons (negative charge) and that certain semiconductors contain positive mobile charges. In 1980, von Klitzing discovered that the relation between the external electric potential and the Hall current is quantized in strong magnetic fields — the conductance,  $I_{Hall}/V$ , comes in units of  $e^2/h$ . Von Klitzing was awarded the 1985 Nobel Prize for his discovery of the Quantum Hall Effect (QHE). There has been a tremendous amount of work on this subject over the past 20 years. New effects — including the *fractional* QHE — have been discovered, and a decent treatment of the subject would fill a course. In Quantum Physics III I would like to explain the origins of the effect — as an extension of the Landau problem — under ideal circumstances. First I will review the ordinary Hall Effect (though I will not assume you have seen it before). Next I will solve an idealized quantum problem: the Schrödinger equation for electrons propagating in the  $xy$ -plane with a magnetic field normal to the plane and an electric field in the  $y$  direction. This will lead

to quantization of the Hall conductance provided we make some simplifying assumptions about the structure of the material in which the electrons propagate. As usual in condensed matter physics, after solving an idealized problem I will have to return to the real world of actual materials and explain why the results of the idealized analysis survive unscathed. Much of my presentation relies heavily on the introductory sections of the review *The Quantum Hall Effect*, by R. E. Prange and S. M. Girvin (Springer-Verlag, Berlin, 1987).

## 4.1 The ordinary Hall effect and the relevant variables

First let us review the ordinary Hall effect. A strip of conductor lies in the  $xy$ -plane. A constant and uniform electric field,  $\vec{E}$ , points in the  $y$ -direction. A constant and uniform magnetic field,  $\vec{B}$ , is oriented normal to the  $xy$ -plane.

First consider the case where  $\vec{B} = 0$ . Mobile charge carriers\*\* with charge  $q$  accelerate in response to  $\vec{E}$ , but suffer random, redirecting collisions with ions. An elementary argument (presented in 8.02?) leads to the conclusion that the electrons develop a *drift velocity*  $\vec{v} = q\vec{E}\tau_0/m$ , where  $\tau_0$  is the average time between collisions. These drifting charges generate a current density (charge per unit time per unit length in the  $xy$ -plane),  $\vec{j} = qn\vec{v}$ , where  $n$  is the density of charge carriers (per unit area). The result is a current density linearly proportional to the impressed electric field,

$$\vec{j} = \frac{nq^2\tau_0}{m}\vec{E}. \quad (4.72)$$

The constant of proportionality relating  $\vec{j}$  and  $\vec{E}$  is the *conductivity*, and has units (in two dimensions)  $[j]/[E] = \ell/t = [\text{velocity}]$ . This is simply Ohm's law with conductivity,  $\sigma_0 = nq^2\tau_0/m$ . It is useful to define the resistivity by the relation  $\vec{E} = \rho\vec{j}$  (which is the local analog of  $V = IR$ ), in which case  $\rho_0 = 1/\sigma_0$ . It is also useful to think of the resistivity (and conductivity) as a matrix relating the vector  $\vec{j}$  to the vector  $\vec{E}$ . In this simple case, the matrix is diagonal,

$$\rho = \begin{pmatrix} \rho_0 & 0 \\ 0 & \rho_0 \end{pmatrix}. \quad (4.73)$$

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\*\*Electrons, for our case, but to keep track of signs we consider the charge carriers arbitrary with charge  $q$ .

When the magnetic field is turned on, the mobile charges respond to the an “effective” electric field arising from the combined electric and magnetic fields according to the Lorentz force law,

$$\vec{F} \equiv q\vec{E}_{\text{eff}} = q\vec{E} + q\frac{\vec{v}}{c} \times \vec{B}. \quad (4.74)$$

The current comes from the charge carriers drift in response to  $\vec{E}_{\text{eff}}$  and is therefore given by  $\vec{j} = \sigma_0 \vec{E}_{\text{eff}}$ . Also,  $\vec{v} = \vec{j}/nq$ , so (4.74) can be rewritten as

$$\vec{j} = \sigma_0 \vec{E} + \frac{\sigma_0}{nqc} \vec{j} \times \vec{B}. \quad (4.75)$$

The current is no longer only parallel to  $\vec{E}$ : Because of the second term in (4.75) it develops a component perpendicular to  $\vec{E}$ . This is most conveniently summarized in terms of a resistivity matrix,  $\vec{E} = \rho \vec{j}$ , which is no longer diagonal. From eq. (4.75) we have

$$\vec{E} = \frac{1}{\sigma_0} \vec{j} + \frac{1}{nqc} \vec{B} \times \vec{j}. \quad (4.76)$$

We take the magnetic field  $\vec{B} = -\hat{e}_3$  and obtain,

$$\rho = \begin{pmatrix} \rho_0 & -\frac{B}{nqc} \\ \frac{B}{nqc} & \rho_0 \end{pmatrix}. \quad (4.77)$$

Note that an electric field in the  $y$ -direction gives rise to a current density in the  $x$ -direction (and vice versa). This *Hall current* is easy to observe and depends on the sign of the charge carriers, because the off diagonal elements of the matrix  $\rho$  depend on the sign of  $q$ . In contrast, the normal resistivity depends only on  $q^2$ . This is the stuff of undergraduate physics labs. This *Hall resistivity* describes the behavior of realistic conductors over a wide range of conditions. Surprisingly the behavior of a system of electrons described by Schroedinger’s equation subject to the same external fields is very different.

## 4.2 Electrons in crossed electric and magnetic fields

### 4.2.1 Setting up the problem

In this section we ignore all the complexities of a physical conductor — electron ion interactions, thermal effects, impurities, and so forth — and

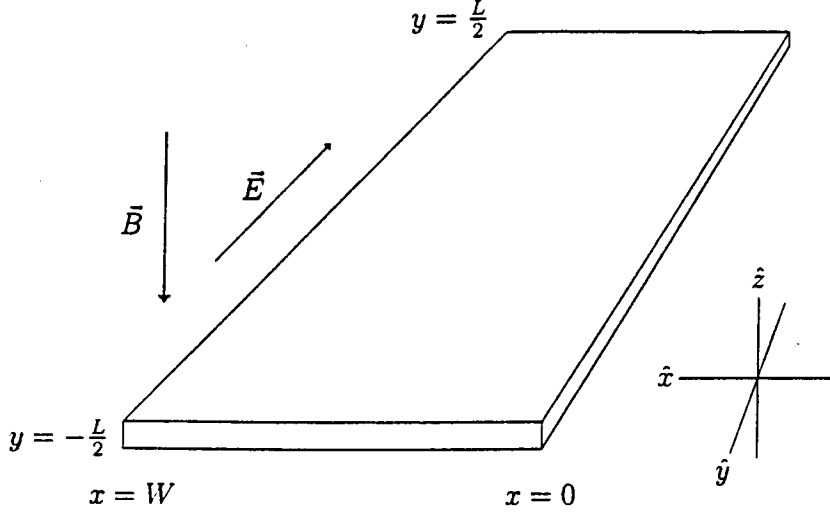


Figure 2: Idealized Hall effect system.

consider the idealized problem of a gas of electrons moving in the  $xy$ -plane subject to an electric field,  $\vec{E} = -E_0\hat{e}_2$ , in the negative  $y$ -direction, and a magnetic field,  $\vec{B} = -B_0\hat{e}_3$  in the negative  $z$ -direction. We assume that the conductor forms a strip with  $0 < x < W$  and study a section between  $y = -L/2$  and  $y = L/2$ . (See Figure 2.) We assume that the magnetic field is sufficiently strong that only one of the two electron spin states is of interest. The other is promoted to higher energy by the dipole interaction energy,  $\mu B_0$ .

To construct the Hamiltonian we need an electrostatic potential to produce  $\vec{E}$  —  $\Phi = E_0y$  — will do; and a vector potential to produce  $\vec{B}$  —  $\vec{A} = B_0y\hat{e}_1$ . Note that we have chosen a different vector potential than we used in our solution to the Landau problem. This choice is more convenient here, but the physics cannot depend on it. The Hamiltonian is given by,

$$\mathcal{H} = \frac{1}{2m} \left[ \left( -i\hbar \frac{\partial}{\partial x} - \frac{eB_0y}{c} \right)^2 - \hbar^2 \frac{\partial^2}{\partial y^2} \right] + eE_0y. \quad (4.78)$$

As usual, it is convenient to introduce some scaled variables. We define the

Larmor frequency  $\omega_L = eB_0/mc$  as usual, and introduce the natural length scale of the Landau problem,  $\ell_0 \equiv \sqrt{\hbar c/eB_0}$ . If we now scale the dimensional factors out of (4.78) we obtain,

$$\mathcal{H} = \frac{\hbar\omega_L}{2} \left[ \left(-i\frac{\partial}{\partial\xi} - \eta\right)^2 - \frac{\partial^2}{\partial\eta^2} + 2\alpha\eta \right], \quad (4.79)$$

where

$$\begin{aligned} \xi &= x/\ell_0 \\ \eta &= y/\ell_0 \\ \alpha &= eE_0\ell_0/\hbar\omega_L. \end{aligned} \quad (4.80)$$

$\xi$  and  $\eta$  are scaled coordinates and  $\alpha$  measures the energy scale of electric relative to magnetic effects.

We now take  $(\xi, \eta, p_\xi, p_\eta)$  to be our canonical variables, so

$$\mathcal{H} = \frac{\hbar\omega_L}{2} \left[ (p_\xi - \eta)^2 + p_\eta^2 + 2\alpha\eta \right], \quad (4.81)$$

where  $p_\xi = -i\partial/\partial\xi$  and  $p_\eta = -i\partial/\partial\eta$ .

#### 4.2.2 Eigenstates and eigenenergies

It is quite straightforward to find the eigenenergies and eigenstates of this Hamiltonian. First note that  $p_\xi$  is a constant of the motion,  $[p_\xi, \mathcal{H}] = 0$ . We denote the eigenvalue of  $p_\xi$  by  $k$ . An eigenstate of  $p_\xi$  can be written in coordinate space as,

$$\begin{aligned} \psi(\xi, \eta) &= e^{ik\xi} \varphi(\eta, k) \text{ with} \\ \mathcal{H}(\eta) \varphi(\eta, k) &= \frac{\hbar\omega_L}{2} \left[ p_\eta^2 + 2\alpha\eta + (k - \eta)^2 \right] \varphi(\eta, k) \\ &= \frac{\hbar\omega_L}{2} \left[ p_\eta^2 + (\eta - k + \alpha)^2 + 2\alpha k - \alpha^2 \right] \varphi(\eta, k). \end{aligned} \quad (4.82)$$

This is simply a one-dimensional harmonic oscillator centered at  $\eta = k - \alpha$  with eigenenergies shifted by  $\hbar\omega_L(\alpha k - \alpha^2/2)$ ,

$$\mathcal{E}_n(k) = \hbar\omega_L \left( n + \frac{1}{2} \right) + \hbar\omega_L \left( \alpha k - \frac{\alpha^2}{2} \right) \quad (4.83)$$

The associated wavefunction,  $\varphi_n(\eta, k)$  is a gaussian (multiplying a Hermite polynomial) strongly localized around  $\eta = k - \alpha$ ,

$$\varphi_n(\eta, k) = \exp -\frac{1}{2}|\eta - k + \alpha|^2 H_n(\eta - k + \alpha). \quad (4.84)$$

The continuous variable  $k$  labels the degeneracy of the Landau levels if  $\vec{E} = 0$ . When  $\vec{E} \neq 0$ , the highly degenerate Landau level spreads out into a band with energies that depend on  $\vec{E}$  through  $\alpha$  as displayed in eq. (4.83).

### 4.2.3 Degeneracies

When  $\vec{E} = 0$  this problem reduces to motion in a constant magnetic field,  $B_0$ , a problem we have just solved, albeit with a different choice for the vector potential  $\vec{A}$ . We can use our earlier solution to help us learn how to count and label the states in the case  $\vec{E} \neq 0$ .

First consider  $\vec{E} = 0$ . When we studied Landau levels we learned to expect one state in each energy level in an area containing one quantum of flux,  $\Phi_0 = hc/e$ . So we expect each Landau level to be  $N$ -fold degenerate, where  $N = \Phi/\Phi_0$ , and  $\Phi$  is the flux,  $\Phi = LWB_0$ , passing through the region  $0 \leq x \leq W$  and  $-L/2 \leq y \leq L/2$ . We can relate this degeneracy to the allowed values of  $k$ , which labels the degeneracy in eq. (4.83). From (4.84) with  $\alpha = 0$  we see that the state with “momentum”  $k$  is localized at  $\eta = k$  or  $y = k\ell_0$ . Thus the states in the region  $-L/2 \leq y \leq L/2$  correspond to a definite range of  $k$ ,

$$-\frac{L}{2\ell_0} \leq k \leq \frac{L}{2\ell_0}. \quad (4.85)$$

We can associate an interval in  $k$  with each state by noting that  $N$  states must fit into the  $k$ -range given by (4.85) —

$$\Delta k = \frac{L}{\ell_0 N}. \quad (4.86)$$

If we substitute  $N = \Phi/\Phi_0$ , we obtain,

$$\Delta k = \frac{2\pi\ell_0}{W}. \quad (4.87)$$

The allowed values of  $k$  can therefore be labelled by an integer,  $p$ ,

$$k_p = \frac{2\pi\ell_0}{W} p + \zeta \quad (4.88)$$

for  $-LW/4\pi\ell_0^2 \leq p \leq LW/4\pi\ell_0^2$ . The constant  $\zeta$  cannot be determined by what we have done so far.

To summarize: For  $\vec{E} = 0$  we have found that the known degeneracy of the Landau levels quantizes the allowed values of the “momentum”,  $k$ , according to the rule (4.88). Referring back to the wavefunction (4.82), we find that (4.88) amounts to a periodicity requirement,  $\psi(x=0, y) = \psi(x=W, y)$  up to the phase  $\zeta W$ . The phase  $\zeta W$  plays no role in the physics, so we set  $\zeta$  to zero henceforth.

Now we return to the case of interest, namely  $\alpha \neq 0$ . We assume that the wavefunction remains periodic in  $x$ :  $\psi(x=0, y) = \psi(x=W, y)$ , and use the  $y$ -dependence to find the range of  $k$ . For  $\alpha \neq 0$ , the eigenstates are centered at  $\eta = k - \alpha$ . This translates into the quantization rule  $k = (2\pi\ell_0/W)p$  with

$$-\frac{LW}{4\pi\ell_0^2} + \frac{\alpha W}{2\pi\ell_0} \leq p \leq \frac{LW}{4\pi\ell_0^2} + \frac{\alpha W}{2\pi\ell_0}. \quad (4.89)$$

The only effect of the electric field is to shift the allowed values of the “momentum”  $k$ .

Let us summarize our solution to the problems of electrons propagating in constant crossed electric and magnetic fields —

- The states are labeled by quantum numbers  $k$  and  $n$ ,

$$\psi(\xi, \eta) = \sqrt{\frac{\ell_0}{W}} e^{ik\xi} \exp -\frac{1}{2} |\eta - k + \alpha|^2 H_n(\eta - k + \alpha) \quad (4.90)$$

where we have introduced a factor  $\sqrt{\ell_0/W}$  so that the wavefunction is normalized to unity over the rectangle of area  $LW$ . (We assume  $\exp(-\eta^2/2)H_n(\eta)$  is normalized in  $\eta$ .)

- The eigenenergies are

$$\mathcal{E}_n(k) = \hbar\omega_L(n + \frac{1}{2}) + \hbar\omega_L(\alpha k - \frac{\alpha^2}{2}). \quad (4.91)$$

- The quantum numbers  $n$  and  $k$  range over the values,

$$\begin{aligned} n &= 0, 1, 2, \dots \\ -\frac{L}{2\ell_0} + \alpha &\leq k \leq \frac{L}{2\ell_0} + \alpha. \end{aligned} \quad (4.92)$$

- The degeneracy of the Landau levels is broken by the electric field. Each Landau level breaks up into a “band” of very closely spaced levels. The width of the band is  $\Delta\mathcal{E} = \hbar\omega_L\alpha L/\ell_0 = eE_0L$ , which is just the change in the classical electrostatic energy of the electron over the length of the conductor. As long as the magnetic field is strong we can assume that the now-smeared-out Landau levels remain well separated from one another.
- The other effect of the electric field is to shift the average “momentum” of the electrons in the Landau bands from *zero* to  $\alpha$ . We shall now see that this shift has the effect of producing the classical Hall current — with a quantum twist...

#### 4.2.4 The Hall current

Now let us calculate the current that flows in the  $x$ -direction in response to the electric field in the  $-y$  direction. The electric current carried by a quantum gas of electrons is its probability current multiplied by the electric charge  $q = -e$ . As you show on a problem set, the expression for the probability current density is

$$\vec{j} = \frac{\hbar}{m} \text{Im} \psi^* \vec{\nabla} \psi - \frac{e}{mc} \vec{A} \psi^* \psi \quad (4.93)$$

In the gauge we are using,  $A_y = 0$ . Given this, and given the fact that the harmonic oscillator wavefunctions,  $H_n$ , are real, we conclude that there is *no current in the  $y$  ( $\hat{e}_2$ ) direction*. This is remarkable since  $\hat{e}_2$  is the direction of the external electric field! The  $x$ -dependence of  $\psi(x, y)$  is complex, so a current in the  $\hat{e}_1$  direction may exist. Substituting (4.90), multiplying by  $-e$ , we obtain a current density in the  $\hat{e}_1$  direction associated with the state labeled by  $n$  and  $k$ . This current density depends on the details of the harmonic oscillator wavefunctions. If we integrate over  $y$  from  $-L/2$  to  $L/2$  we obtain a simple expression for the current in the  $\hat{e}_1$  direction from an electron with “momentum”  $k$  in the  $n^{\text{th}}$  Landau level,

$$I_H(n, k) = -\frac{e\hbar k}{m\ell_0 W}. \quad (4.94)$$

where the subscript  $H$  on  $I$  reminds us that this is the Hall current. Note that the contribution of the  $\vec{A}\psi^*\psi$  term in the current density (4.93) is odd in  $y$  and so vanishes once we integrate over  $y$ .

Now let us suppose that *all states in a given Landau band are filled* and calculate the associated current. This means that we sum (4.94) over the allowed range of  $k$ . For  $\alpha = 0$  the negative and positive values of  $k$  cancel and we get  $\vec{j} = 0$  for  $\alpha = 0$ . For  $\alpha \neq 0$  we replace the sum over  $p$  by an integral,

$$I_H(n) = -\frac{e\hbar}{m\ell_0 W} \int_{p_-}^{p_+} k(p) dp \quad (4.95)$$

where  $p_{\pm}$  are the limits on  $p$  given in eq. (4.89). This yields,

$$I_H(n) = -\frac{e^2}{h} E_0 L \quad (4.96)$$

corresponding to a Hall conductance of  $\sigma = e^2/h$  *independent of the size of the sample ( $L$  and  $W$ ) and  $n$  — the label of the Landau band*. If  $N$  Landau bands are filled then the conductance is  $-Ne^2/h$ . In the matrix notation of the previous section, we have found that the idealized quantum problem leads to a purely *off-diagonal* conductance matrix

$$\sigma = \begin{pmatrix} 0 & -\frac{Ne^2}{h} \\ +\frac{Ne^2}{h} & 0 \end{pmatrix}, \quad (4.97)$$

The resistance matrix is the inverse of  $\sigma$ .

To summarize: *If* the idealized quantum treatment is justified, the conductance is purely off-diagonal: an external field in the  $\hat{e}_2$  direction leads to a current in the  $\hat{e}_1$  direction; and *if*  $N$  Landau bands *are filled* then the off diagonal (Hall) conductance is quantized in multiples of  $e^2/h$ .

#### 4.2.5 A description of the integer quantum Hall effect

Now it is necessary to translate the rather abstract calculation we have just completed into a description of the effect observed by von Klitzing and others. I don't have the time or expertise to do justice to the richness of the phenomena in realistic conductors, but I will try to sketch the physics in a somewhat idealized situation.

Figure 4 shows the results of a measurement of the Hall resistance (on the left vertical axis) and the longitudinal resistance (on the right vertical axis) as a function of  $n\hbar c/eB_0$ , measured on an idealized sample at low temperature. Two effects stand out: (1) the Hall conductance ( $1/R_H$ ) is

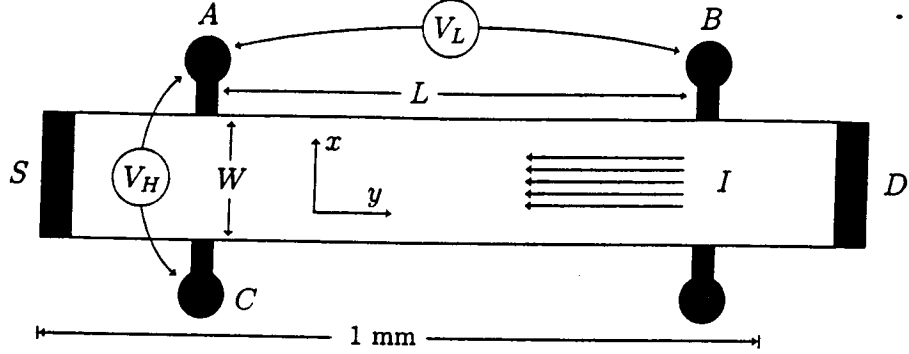


Figure 3: (From R. E. Prange and S. M. Girvin, *The Quantum Hall Effect*, Springer-Verlag, Berlin, 1987.

quantized in integer multiples of  $e^2/h$  over *ranges* of magnetic field strength. The plateaus in  $1/R_H$  are very flat and separated by steep increases as  $1/R_H$  grows from one integer value to another. (2) The longitudinal resistance, on the other hand, vanishes when  $R_H$  is constant, and is finite when  $R_H$  is varying. The longitudinal resistance depends on the geometry of the sample as it does for normal metals under normal conditions. The possible values of the Hall resistance, on the other hand, are independent of the geometry of the sample. It depends on the sample only through the magnetic field strength,  $B_0$  and the charge density,  $n$ . Note that a smooth line interpolating through the steps in  $R_H$  gives the *classical* relationship  $1/R_H = nec/B_0$  as expected from the classical analysis at the beginning of this section (see eq. (4.77)).

To understand Fig. 4 it is necessary to figure out what is happening to the electron spectrum as  $n\hbar c/eB_0$  is increased. Since electrons obey the Pauli exclusion principle, only two electrons (one of each spin projection) can be placed in each spatial state. For  $B_0 = 0$  the states available to the electrons form (essentially) a continuum. For  $B_0 \neq 0$  the continuum breaks up into Landau levels separated by gaps. Each Landau level can hold one electron in an area corresponding to a single flux quantum. That area is given by  $\hbar c/eB_0$ . The number of electrons that can be accommodated in a given Landau level grows as  $B_0$  increases. When  $B_0$  is extremely large, all the electrons can be placed in the lowest Landau level. For large  $B_0$ , the

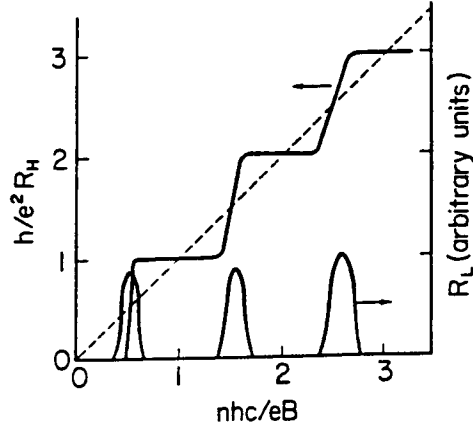


Figure 4: Longitudinal and transverse resistance for the quantum Hall effect. (From Prange and Girvin.)

splitting between spin up and spin down electron levels also becomes large, so we can ignore the higher spin state. The ratio of the number of electrons per unit area,  $n$ , and the number per unit area per Landau level,  $eB_0/hc$ , gives us the number of Landau levels that must be occupied as a function of  $B_0$  (at fixed  $n$ ),

$$N(B_0) = nhc/eB_0, \quad (4.98)$$

which is exactly the independent variable in Fig. 4.

We know from the analysis of the previous section that the Hall conductance is quantized when a Landau level is *exactly* full. Fig. 4 together with our interpretation of the variable  $nhc/eB_0$  seems to be saying that  $1/R_H$  behaves as though the Landau level were exactly full for a range of  $B_0$ .

The key to understanding the integer quantum Hall effect is the role of *impurities*. We have treated the electrons as an ideal gas, free to orbit in Landau levels. Real experiments are done using almost-two-dimensional electron systems called “inversion layers”, in which the electrons are essentially free to move in the  $xy$ -plane but are localized in  $z$  at a planar interface between one type of semi-conductor and another. In any real system, there are impurities. This means that while many electron energy levels are *delocalized* as in the ideal case — allowing them to move through the two-dimensional system and respond to external  $\vec{E}$  and  $\vec{B}$  fields, other energy levels remain pinned, *ie.*

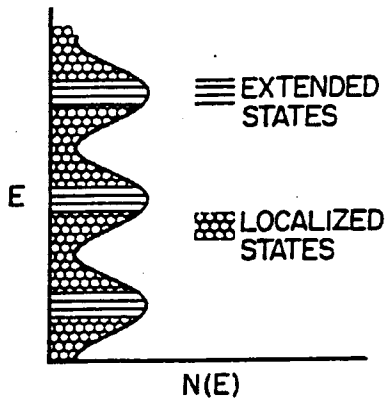


Figure 5: Localized and extended states in a “realistic” two-dimensional electron gas. (From Prange and Girvin.)

*localized*, around impurities. Electrons that fill these localized levels do not conduct current. A picture of the energy levels in a realistic two-dimensional electron gas is shown in Fig. 5: localized states in between bands (in this case Landau levels) of delocalized states. The striking appearance of the integer quantum Hall effect can be understood by considering the sequential filling of localized and delocalized states as a function of  $1/B_0$ .

We begin with very large  $B_0$ , at the far left of Fig. 4. The lowest Landau level is only partially filled. Electrons in a partially filled Landau level give rise to a Hall conductance that is a fraction (the “filling fraction”) of the quantum  $e^2/h$ . As  $B_0$  decreases, the Landau level fills. The conductance is exactly  $e^2/h$ . As  $B_0$  decreases further electrons are forced into higher *localized* levels between the first and second Landau levels. These electrons do not conduct, so the Hall conductance stays fixed at  $e^2/h$ . When  $B_0$  decreases still further, the second Landau level quickly fills and the conductance rises to  $2e^2/h$ . Succeeding intervals of filling localized and delocalized levels gives rise to the step-like pattern shown in Fig. 4. Eventually, at low magnetic field strength, the steps smooth out to follow the interpolating dashed line that marks the classical Hall resistance.

The explanation of the behavior of the longitudinal resistance also depends crucially on the existence of impurities. In the last section we found

that electrons in Landau levels do not contribute to the longitudinal conductivity of a sample. This idealization breaks down in the presence of electron scattering from impurities. Such scattering generates a normal, longitudinal resistance when Landau levels are partially full. However, when a given Landau level is exactly full, the electrons have no unoccupied levels into which to scatter and the longitudinal resistance vanishes. In the domains of  $B_0$  where the localized states are filling the situation remains unchanged — the localized electrons do not participate. When a Landau level again begins to fill, scattering again generates a longitudinal resistance.

We have only scratched the surface of the myriad of phenomena associated with the behavior of materials in the presence of magnetic fields. Many aspects of these systems are studied by condensed matter theorists and experimenters in the MIT Physics Department. For further readings we recommend the book by Prange and Girvin.