Prof. Jaffe's notes on: Scattering Theory Intro (so you are familiar with his notation) and the Partial Wave Analysis

SCATTERING THEORY IN 3 DIMENSIONS

ELEMENTARY KINEMATICS OF SCATTERING

The small size of quantum systems compared to macroscopic objects makes scattering experiments one of the only available tools of analysis. Typically a beam of particles of known type are accelerated, collimated and filtered to a reasonably narrow momentum band. We will assume the beam has a sharp momentum $\vec{k}$, although this would require the beam particles to be unlocalizable in space which is not realistic. [More @ this later]

The beam impinges on a target of density ρ and the beam particles scatter. Various things can happen:

- Elastic scattering — $a+b \to a+b$ — same particles come out without production of new particles or internal excitation
- Inelastic scattering — anything else. Either internal excitation $a+b \to a+b^*$, for example, or production: $a+b \to a+b+c$
- Absorption — $a+b \to (\text{not } a) + \text{anything}$

What is measured is a number of particles per unit time which do something. This is compared to the number of particles per unit time and per unit area impinging on the target:

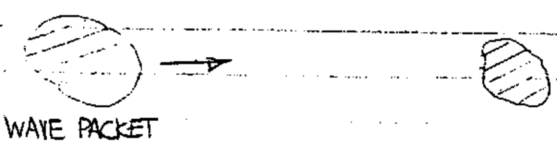
$$\frac{\frac{dN}{dt}(\text{outcome})}{\frac{dN}{dt\,dA}(\text{incident})} \equiv \sigma(\text{outcome})$$

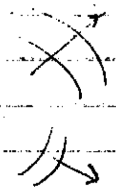
Dimensions of area. Called "cross section for outcome"

Examples:
- σ_{elastic} — cross section for elastic scattering
- $\sigma_{\text{inelastic}}$ — " " " inelastic "
- σ_{TOT} — total cross section
- σ_{ABS} — absorption cross section
- $\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}}$ — elastic differential cross section, per element of solid angle.

Physical picture:

INITIAL:



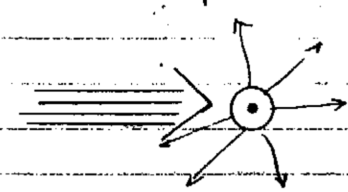
FINAL : 

IDEALIZATION: Spinless particle scattering from a static potential.

→ Suppose wave packet is large compared to the size of the potential. That means that the spread in momentum is small compared to the characteristic energies required to excite the potential:

$$\Delta p << \hbar/R$$

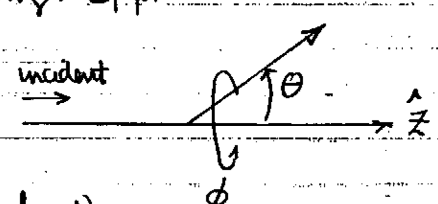
In this case during the interaction time the incident wave packet can be reasonably approximated as a plane wave. This plane wave will produce an outgoing scattered wave



At large distances from the potential both the incident and scattered waves are solutions to the free schroedinger equation.

$$\psi_{inc} \propto e^{ikz - iEt}$$ taking incident wave along the z-direction

Scattered wave originates from scattering potential which we place at origin, $\vec{x} = 0$. It must be outgoing in r: So what is form of Schroedinger eq for large r and energy E:



$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \qquad \frac{2mE}{\hbar^2} = k^2$$

$$\left(-\frac{1}{r}\frac{d^2}{dr^2}r + \frac{1}{r^2}L^2\right)\psi = k^2\psi \qquad \text{let } r\psi = u(r)\, f(\theta,\phi)$$

$$-u''f + \frac{1}{r^2}L^2 uf = k^2 uf \qquad \text{as } r\to\infty \quad u'' + k^2 u = 0 \to u = e^{\pm ikr}$$

Choose e^{+ikr} so wave is outgoing: $e^{ikr - iEt}$
but $f(\theta,\phi)$ is arbitrary

so $$\psi_{scatt} \sim \frac{f(\theta,\phi)}{r} e^{ikr - iEt} \quad \text{as } r\to\infty.$$

Now we can replace wave packet picture by a time independent scattering formalism provided we recognize that incident wave and outgoing wave do not interfere — even though, if we just followed the rules, they would. The reason they don't is that they are wave packets and do not appear at the same place at the same time.

This enables us to REPLACE THE SCATTERING PROBLEM WITH A SIMPLE WAVE MECHANICS PROBLEM:

- Solve TIME INDEPENDENT SCHROEDINGER EQ.
- Subject to the boundary condition $\lim_{r\to-\infty} \psi(r) \sim e^{ikz} + \frac{f(\theta,\phi)}{r} e^{ikr}$

These are called scattering state boundary conditions.

This simple case illustrates a more general point: scattering can be handled by choosing appropriate, physically motivated boundary conditions. Generally incoming particles as $t\to-\infty$ and outgoing particles as $t\to+\infty$.

$f(\theta,\phi)$ ≡ SCATTERING AMPLITUDE

CALCULATE CROSS SECTION:

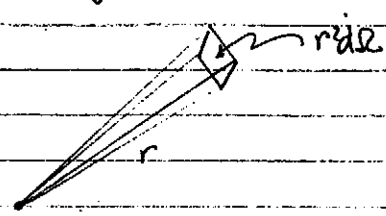
INCOMING FLUX: $\vec{S}_{in} = \frac{\hbar}{2mi}\left(\psi^*\vec{\nabla}\psi - (\vec{\nabla}\psi^*)\psi\right) = dN/dA\,dt$

$$\text{for } \psi = e^{ikz} = \frac{\hbar k}{m}\hat{z} = v\hat{z} \qquad (\vec{v} = \vec{p}/m)$$

OUTGOING FLUX: $\vec{S}_{scat} = \frac{\hbar}{2mi}(2ik)\frac{\hat{r}}{r^2}|f|^2 = \frac{v\hat{r}}{r^2}|f(\theta)|^2$ (note in differentiating, ignore $1/r^2$ rel. to $1/r$)

What is flux across a small sector of a sphere of surface area $r^2 d\Omega$

$$\vec{S}_{scat}\cdot\hat{r}\,dA = v\,|f(\Omega)|^2 d\Omega$$



$$\frac{\vec{S}_{scat}\cdot\hat{r}\,dA}{|\vec{S}_{inc}|} = |f(\Omega)|^2 d\Omega \equiv \frac{d\sigma}{d\Omega}d\Omega$$

= Cross section to scatter into solid area element $d\Omega$ between θ and $\theta+d\theta$, ϕ and $\phi+d\phi$

$$\boxed{\frac{d\sigma_{scat}}{d\Omega} = |f(\theta,\phi)|^2}$$

UNITARITY What goes in must come out.

When we calculated outgoing flux we made a mistake in the extreme forward direction because at $\theta = 0$ the unscattered incident beam can interfere with the scattered wave. This is not a failure of the time independent treatment, it occurs for wavepackets too.

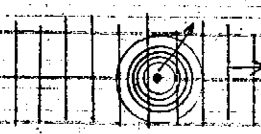
We must have made a mistake because probability isn't conserved!

IN $\quad \frac{dN}{dA\,dt} = v \qquad$ number incident across plane $\perp$ to z direction

OUT $\quad \frac{dN}{dA\,dt} = v \qquad$ number out across plane $\perp$ z direction (from e^{ikz})

PLUS $\quad \int d\Omega\, |f|^2$ scattered

Recalculate outgoing flux more carefully near $\theta = 0$ looking for interference



$$S_{out} = \frac{\hbar}{2mi}\left(\psi^{+}\vec{\nabla}\psi - h.c.\right)$$

$$\psi^{+}\vec{\nabla}\psi = \left(e^{-ikz} + \frac{f^*}{r}e^{-ikr}\right)\left(ik\hat{z}\,e^{ikz} + \frac{ik\hat{r}f}{r}e^{ikr} + \dots\right)$$

$$= ik\left(\hat{z} + \frac{\hat{r}}{r^2}|f(\theta)|^2 + \frac{\hat{z}f^*}{r}e^{ik(z-r)} + \frac{\hat{r}f}{r}e^{ik(r-z)} + \dots\right)$$

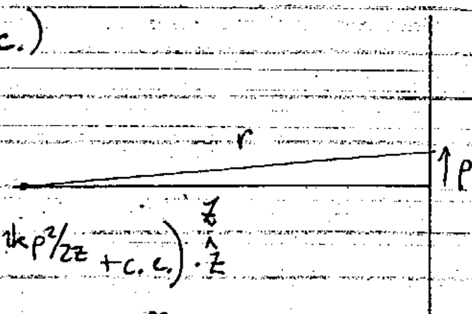
When $r \to \infty$ $e^{ik(z-r)}$ averages to zero except very close to $\theta = 0$.

Calculate flux across plane $\perp$ to z-axis very far from scattering center from interference term

$$\vec{S}_{interference} = \frac{v}{2}\left(\frac{\hat{z}f^*}{r}e^{ik(z-r)} + \frac{\hat{r}f}{r}e^{ik(r-z)} + c.c.\right)$$

$r = z + \rho^2/2z$

$\hat{r} \approx \hat{z}$



$$\int \vec{S}_{int}\cdot\hat{z}\,dA = \frac{v}{2}\int_0^{2\pi}d\phi\int_0^{\infty}\rho\,d\rho\left(\frac{\hat{z}}{z}f^*e^{-ik\rho^2/2z} + \frac{\hat{z}f}{z}e^{ik\rho^2/2z} + c.c.\right)\cdot\hat{z}$$

$$= 2\frac{v}{z}\,2\pi\int_0^{\infty}\frac{d\rho^2}{2}\,\mathrm{Re}\left[f e^{ik\rho^2/2z}\right] = 2\frac{v\pi}{z}\,\mathrm{Re}\int_0^{\infty}dy\; e^{iky/2z}f(0)$$

$$= 2\frac{v\pi}{z}\,\mathrm{Re}\left[+\frac{2zi}{k}f(0)\right] = -\frac{4\pi v}{k}\,\mathrm{Im}\,f(0)$$

Now we add up all the outgoing flux and compare to incoming flux

$$\underset{\substack{\uparrow \\ \text{plane} \\ \text{wave in}}}{\int \vec{S}_{in} \cdot \hat{z}\, dA} = \underset{\substack{\uparrow \\ \text{plane} \\ \text{wave out}}}{\int v_i\, dA} + \underset{\substack{\uparrow \\ \text{spherical} \\ \text{wave out}}}{v \int d\Omega\, |f(\theta)|^2} - \underset{\substack{\uparrow \\ \text{interference}}}{\frac{4\pi v}{k} \operatorname{Im} f(0)}$$

$$\rightarrow \quad \underset{\substack{\uparrow \\ \text{AMOUNT SCATTERED}}}{\int d\Omega\, |f(\Omega)|^2} = \frac{4\pi}{k} \operatorname{Im} f(0)$$

AMOUNT SCATTERED = AMOUNT REMOVED FROM INCIDENT BEAM.

- Example of, but by no means only consequence of "UNITARITY"
- Bohr, Peierls, Placzek
- Other applications: blue sky & red sunsets

PARTIAL WAVES

In the case that the potential is spherically symmetric, angular momentum conservation simplifies the problem.

APPROACH: Solve Schroedinger equation for energy E.
Match to asymptotic form

General solution:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi = E \psi$$

Match to $\lim_{r \to \infty} \psi(r) = e^{ikz} + \frac{f(\theta,\phi)}{r} e^{ikr}$

Wave equation and incoming wave do not depend on ϕ so we expect

$f(\theta,\phi) \to f(\theta)$ and ψ does not depend on ϕ.

So $\psi = \sum_l i^l (2l+1) R_l(r) P_l(\cos\theta)$ Since $Y_{l0} \propto P_l(\cos\theta)$ is only m-value independent of ϕ.

$$R_l(r) = \frac{u_l(r)}{r}$$

$u_\ell(r)$ obeys the radial equation

$$u_\ell'' + \left(k^2 - U(r) - \frac{\ell(\ell+1)}{r^2}\right) u_\ell = 0$$

Note we will try to use notation that $R_\ell(r)$ is the "true" wave function and $u_\ell(r) = r\,R_\ell(r)$ is the "radial" wavefunction.

SCATTERING AMPLITUDE IS DETERMINED BY ASYMPTOTIC FORM OF W.F. AT LARGE r.

TAKE $r \to \infty$. ASSUME $r^2 U(r) \to 0$ (NOT COULOMB!)
LET $x \equiv kr$

$$u_\ell'' + u_\ell - \frac{\ell(\ell+1)}{x^2} u_\ell = 0$$

This is BESSEL'S EQUATION FOR HALF-INTEGER ORDER.

SOLUTIONS: SPHERICAL BESSEL'S FUNCTIONS (SEE 8.05)

$u_\ell(x) = x\,j_\ell(x)$ $\quad$ $j_\ell(x)$ IS REGULAR AT $x=0$

$\quad = x\,n_\ell(x)$ $\quad$ $n_\ell(x)$ IS SINGULAR AT $x=0$

Spherical Bessel functions are really simple — see sheets of properties of j_ℓ and n_ℓ.

Especially $\quad j_0(x) = \frac{\sin x}{x} \quad j_1 = -\frac{\cos x}{x} + \frac{\sin x}{x^2}$

$\quad n_0(x) = -\frac{\cos x}{x} \quad n_1 = \frac{\sin x}{x} + \frac{\cos x}{x^2}$ etc.

Near $x=0$: $\quad j_\ell(x) \sim x^\ell/(2\ell+1)!!$

$\quad n_\ell(x) \sim -(2\ell-1)!!/x^{\ell+1}$

$x \to \infty$ $\quad j_\ell(x) \sim \frac{1}{x}\sin(x - \ell\pi/2) + \mathcal{O}(1/x^2)$

$\quad n_\ell(x) \sim -\frac{1}{x}\cos(x - \ell\pi/2) + \mathcal{O}(1/x^2)$

We are interested in the asymptotic (large r) behavior of a solution to full schroedinger equation

- with angular momentum ℓ
- regular at origin (only one solution)
- depends only on θ

$$\psi(r,\theta) \xrightarrow[\text{each } \ell]{\text{for}} R_\ell(r)\underbrace{(2\ell+1)i^\ell}_{\text{conventions}} P_\ell(\cos\theta)$$

Asymptotically

$$R_\ell(r) \longrightarrow j_\ell(x) + \beta_\ell n_\ell(x)$$

$$\xrightarrow[x\to\infty]{} \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2} + \delta_\ell(k)\right)$$

$$\hookrightarrow \cos\delta_\ell \sin(x-\ell\pi/2) + \sin\delta_\ell\cos(x-\ell\pi/2)$$

$$\sim \cos\delta_\ell\,(j_\ell(x) - \tan\delta_\ell\, n_\ell(x))$$

so $\beta_\ell = -\tan\delta_\ell$

So for each ℓ, all information is encoded in a single phase $\delta_\ell(k)$. We should be able to understand scattering in terms of $\delta_\ell(k)$.

SCATTERING AMPLITUDE:

Return to $\quad e^{ikz} + \frac{e^{ikr}}{r} f(\theta,\varphi)$

Must understand e^{ikz} in spherical coordinates

Since it is regular at origin e^{ikz} must have expansion in $P_\ell(\cos\theta)j_\ell(kr)$

ANSWER:

$$e^{ikz} = \sum_\ell i^\ell P_\ell(\cos\theta)(2\ell+1)j_\ell(kr)$$

STUDY AS SCATTERING STATE

$$\lim_{r\to\infty} e^{ikz} = \sum_\ell i^\ell(2\ell+1)P_\ell(\cos\theta)\frac{1}{2ikr}\left(e^{ikr-i\ell\pi/2} - e^{-ikr+i\ell\pi/2}\right)$$

TWO EXPONENTIAL TERMS —INTERPRET:

$\frac{e^{-ikr}}{r}$ corresponds to incoming spherical wave with a certain flux

$\frac{e^{ikr}}{r}$ corresponds to outgoing spherical wave with <u>SAME</u> flux.

What modification is possible consistent with flux conservation?

ONLY $e^{+ikr} \longrightarrow e^{i\phi_\ell} e^{ikr}$ ONLY A PHASE MODIFICATION
INCOMING WAVE NOT MODIFIED

DEFINE $\phi_\ell \equiv 2\delta_\ell$ SO

$$\psi_+(r) = \sum_\ell i^\ell (2\ell+1) P_\ell(\cos\theta) \frac{1}{2ikr} \left(e^{2i\delta_\ell + ikr - \frac{\ell\pi}{2}i} - e^{-ikr + i\ell\pi/2} \right)$$

$$= e^{ikz} + \left\{ \sum_\ell \cancel{i^\ell} (2\ell+1) P_\ell(\cos\theta) (e^{2i\delta_\ell} - 1) \cancel{e^{-\ell\pi i/2}} \right\} \frac{e^{ikr}}{r}$$

Compare

$$f(\theta) = \sum_\ell (2\ell+1) P_\ell(\cos\theta) \frac{(e^{2i\delta_\ell} - 1)}{2ik}$$

$$= \sum_\ell (2\ell+1) P_\ell(\cos\theta) f_\ell$$

$$\underline{f_\ell = \frac{1}{2ik}(e^{2i\delta_\ell} - 1)}$$

Note that this is exactly the same definition of δ_ℓ we used before because

$$e^{2i\delta_\ell + ikr - i\ell\pi/2} - e^{-ikr + i\ell\pi/2} \sim e^{i\delta_\ell} \sin(kr - \ell\pi/2 + \delta_\ell)$$

Many equivalent forms for f_ℓ

$$f_\ell = \frac{1}{k} \sin\delta_\ell \, e^{i\delta_\ell}$$

$$= \frac{1}{k} \frac{1}{\cot\delta_\ell - i}$$

$$= \frac{1}{k} \sin\delta_\ell / e^{-i\delta_\ell}$$

$$= \frac{1}{k} \frac{\sin\delta_\ell}{\cos\delta_\ell - i\sin\delta_\ell}$$

$$f(\theta) = \sum_{\ell} (2\ell+1)\, P_\ell(\cos\theta)\, f_\ell$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell} \sin\delta_\ell\, P_\ell(\cos\theta)\right|^2$$

DIFFERENT PARTIAL WAVES INTERFERE

TOTAL CROSS-SECTION : $\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\, P_m(\cos\theta) = 2\frac{\delta_{\ell m}}{2\ell+1}$

$$\sigma_{TOT} = \int d\Omega \frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty} (2\ell+1)\sin^2\delta_\ell$$

DISCUSSION

1. OPTICAL THM: $\operatorname{Im} f(\theta)\Big|_0 = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{k} P_\ell(0)\sin^2\delta_\ell$

$P_\ell(0) = 1$

$$\operatorname{Im} f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty} (2\ell+1)\sin^2\delta_\ell$$

$$= \frac{k}{4\pi}\sigma_{TOT} \quad ✓$$

2. PARTIAL WAVE UNITARITY

$\sigma_\ell = \frac{4\pi}{k^2}\sin^2\delta_\ell \leq 4\pi/k^2$ so total cross section in a given partial wave is bounded.

3. INELASTIC SCATTERING

Suppose incident wave is absorbed or scattered into some other channel. Then scattered wave is diminished

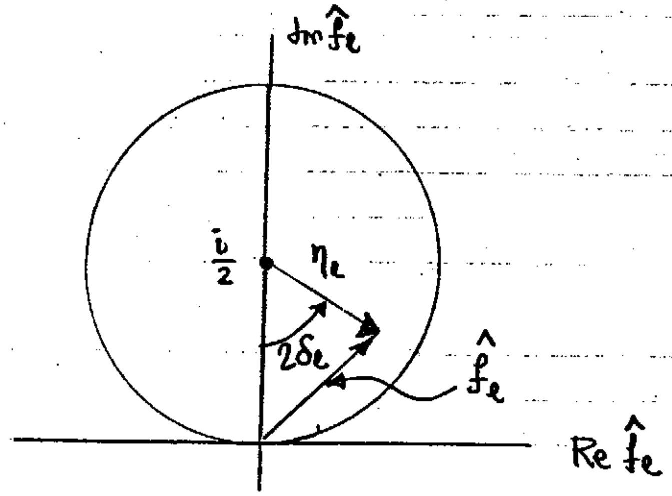
$$e^{ikr - i\ell\pi/2} \longrightarrow \eta_\ell\, e^{2i\delta_\ell + ikr - i\ell\pi/2}$$

with $\eta_\ell \leq 1$ $\quad$ η_ℓ is ELASTICITY

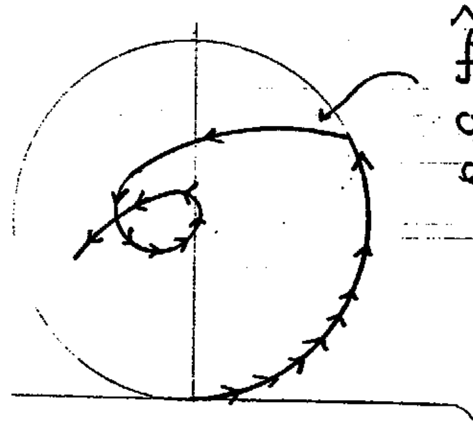
$$f_\ell = \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2ik}$$

f_ℓ IN COMPLEX PLANE – ARGAND DIAGRAM

$$\hat{f}_\ell = k f_\ell = \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2i} = \frac{i}{2} - \frac{i}{2}\eta_\ell e^{2i\delta_\ell}$$



Path of $\hat{f}_\ell$ in complex plane lies on (elastic) or inside (inelastic) the unitarity circle (radius 1/2 centered at $i/2$)



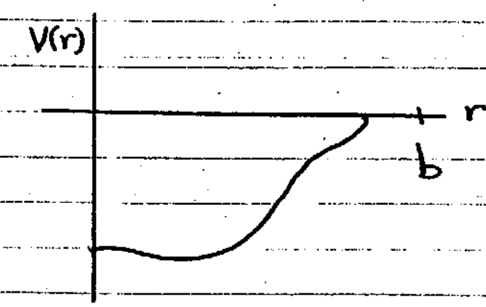
$\hat{f}_\ell(k)$ as a function of k. "Ticks" are equally spaced in k.

Known as Argand diagram of scattering amplitude.

SUPPOSE SOLN TO S.Eq. IS KNOWN. In the form of $R_\ell(r)$. Call this the interior solution.

Want to use $\lim_{r\to\infty} R_\ell(r) = A_\ell\left(j_\ell(kr) - \tan\delta_\ell\, n_\ell(kr)\right)$

Instead of taking $r\to\infty$, choose an $r\equiv b$ beyond the range of $V(r)$ and MATCH THERE



Form log-derivative at b:

$$\rho_\ell = \frac{b\, dR_\ell/dr\big|_b}{R_\ell(b)}$$

equate to external form:

$$= kb\,\frac{\left(j_\ell'(kb) - \tan\delta_\ell\, n_\ell'(kb)\right)}{j_\ell(kb) - \tan\delta_\ell\, n_\ell(kb)}$$

Solve for $\tan\delta_\ell$:

$$\tan\delta_\ell = \frac{x j_\ell'(x) - \rho_\ell j_\ell(x)}{x n_\ell'(x) - \rho_\ell n_\ell(x)} \qquad (*)$$

EXAMPLES

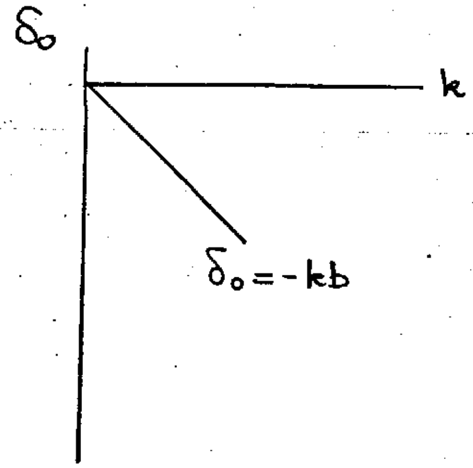
1. HARD SPHERE:

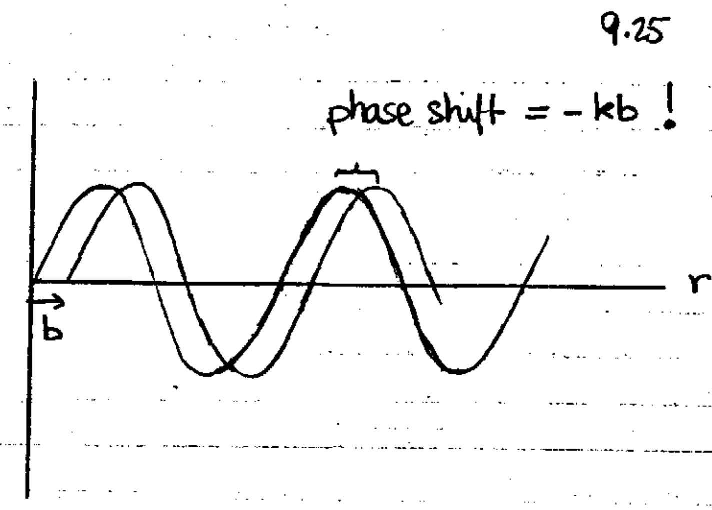
$$\psi = 0 \text{ at } r = b \implies \rho_\ell \to \infty$$

$$\tan\delta_\ell = +j_\ell(kb)/n_\ell(kb)$$

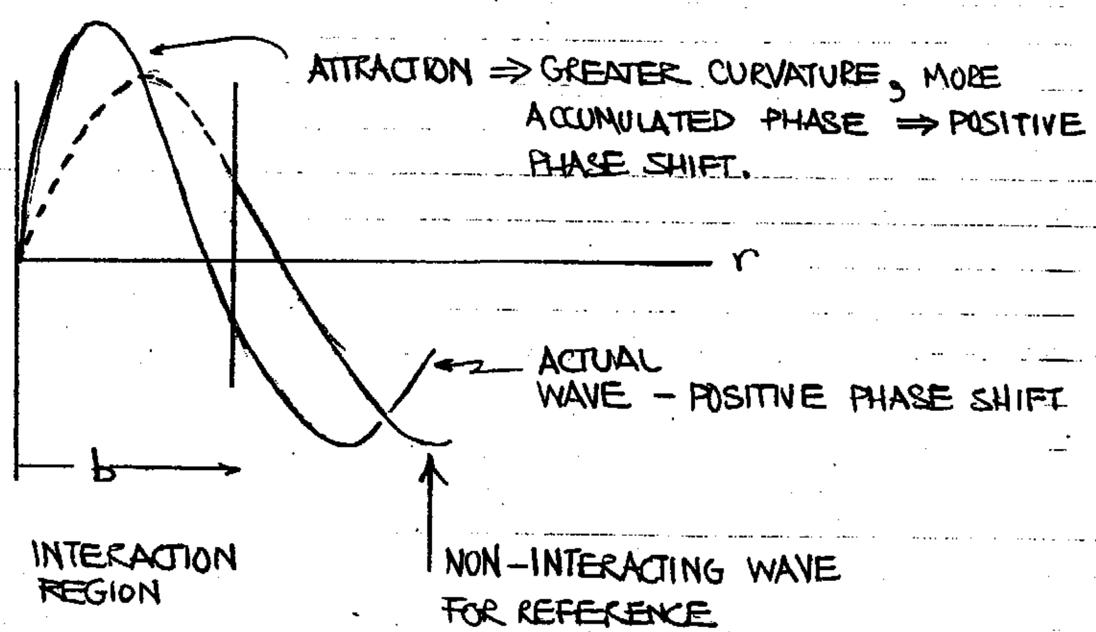
EG. $\ell = 0$ $\qquad \tan\delta_0 = -\sin kb/\cos kb$

$$\delta_0 = -kb$$





2. SIGN OF PHASE SHIFT & ATTRACTION



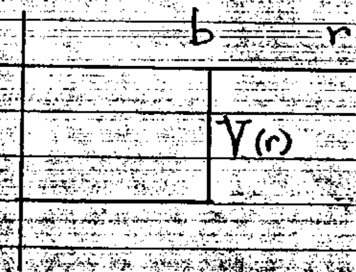
Attraction "pulls" wave fn into interaction region ⇒ +'ve phase shift
Repulsion "pushes" wave fn out ⇒ –'ve phase shift.

NOTE — This argument can be confusing due to 2π ambiguity
a phase shift of $2\pi - \epsilon$ (strong attraction) can be
mistaken for $-\epsilon$ (weak repulsion).

Theorists can avoid ambiguity by starting with $V=0$
and turning on V with a parameter λ. Then δ
varies continuously w/ λ.

3. SQUARE WELL

$$U = -2mV_0/\hbar^2 \equiv -\gamma^2$$



$$R_0(r) = \frac{\sin qr}{qr} \quad \text{(norm arbitrary)}$$

$$\rho_0 = b\left(\frac{q\cos qb}{qb} - \frac{\sin qb}{qb^2}\right) \Big/ \sin qb/qb$$

$$= y\cot y - 1 \quad (y \equiv qb)$$

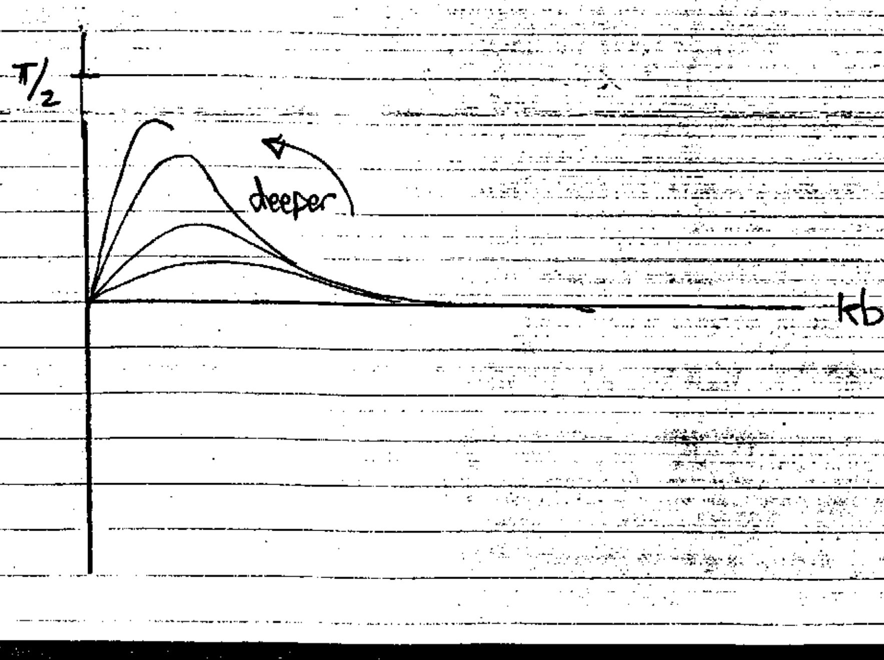
Use (*) from (9.24)

$$\tan\delta_0 = \frac{x\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) - \rho_0 \frac{\sin x}{x}}{x\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right) + \rho_0 \frac{\cos x}{x}} \qquad \text{mult N \& D by } x/\sin x$$

$$\tan\delta_0(x) = \frac{x\cot x - 1 - \rho_0}{x + (1+\rho_0)\cot x} = \frac{x\cot x - y\cot y}{x + y\cot x\cot y}$$

or

$$\underline{x\cot(\delta_0(x) + x) = y\cot y}$$



LOW ENERGY BEHAVIOR OF PHASE SHIFTS

$j_\ell(x) \sim x^\ell/(2\ell+1)!!$

$n_\ell(x) \sim -\frac{1}{x^{\ell+1}}(2\ell-1)!!$

Calculate derivatives and substitute into (*) on (9.24)

$$\tan\delta_\ell(x) \sim \frac{x^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}\;\frac{\ell-\rho_\ell+\theta(x^2)}{\ell+1+\rho_\ell+\theta(x^2)}$$

* UNLESS $\beta_\ell(k=0)$ IS ACCIDENTALLY EQ. TO SOME SPECIAL VALUE,

$$\tan\delta_\ell \sim k^{2\ell+1}$$

ONLY $\ell=0$ AT VERY LOW ENERGIES

$\sigma \rightarrow \sigma_{\ell=0}$ AT LOW ENERGIES

$$\tan\delta_0 \sim kb\left(\frac{-\beta_0}{1+\beta_0}\right) = -ka \qquad a \Rightarrow \text{SCATTERING LENGTH}$$

$$\sigma \rightarrow \sigma_0 = \frac{4\pi}{k^2}(ka)^2 = 4\pi a^2$$

• HARD SPHERE: $\delta_0(k) = -kb$ so scattering from hard sphere is $4\pi b^2$ at low energy
Note $= 4\times$ geometric! DIFFRACTION

• "SQUARE WELL" $\quad kb\cot(kb+\delta_0(k)) = qb\cot qb$

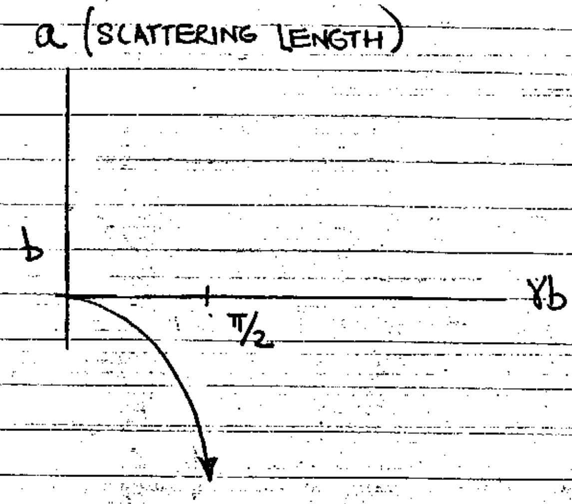
$$\left.\begin{array}{l}\delta \rightarrow -ka\\ q \rightarrow \gamma\end{array}\right\}\text{as } k\rightarrow 0 \qquad \cot\epsilon \sim \frac{1}{\epsilon}$$

$$\frac{b}{b-a} = \gamma b\cot\gamma b$$

- $1 - a/b = \frac{1}{\gamma b} \tan \gamma b$

$$a = b - \frac{1}{\gamma} \tan \gamma b$$

So $a \to \infty$ as $\gamma b \to \pi/2$



This is a special value of well strength – where it first has a bound state. So $a \to \infty$ when $V(r)$ has a bound state at zero energy!

$a = 0$ when $\gamma b = \tan \gamma b \Rightarrow$ Ramsauer-Townsend effect – No scattering!

- PHYSICAL SIGNIFICANCE OF LARGE S-WAVE SCATTERING LENGTH

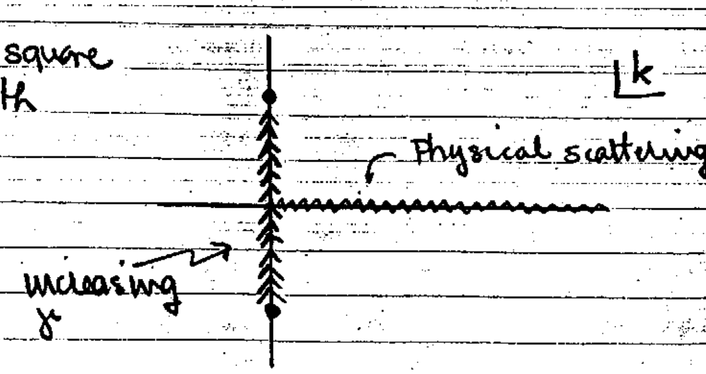
$\tan \delta_0 = -ka + \mathcal{O}(k^2)$ with a anomalously large.

$$f_\ell = \frac{1}{k(\cot \delta - i)} = \frac{1}{k\left(-\frac{1}{ka} - i\right)} = \frac{1}{-ik - 1/a} = \frac{i}{k - i/a}$$

So f_ℓ has a pole at $k_0 = i/a$

1. This is very near physical region because a is large.
2. The extrapolation to locate the pole is reliable because further terms in expansion of $\delta_0(k)$ should not be very important when $|k| \sim 1/a$.

Pole at $k_0 = i/a$ for square well as function of depth



- $\gamma \to 0 \Rightarrow a \to 0$
- γ increasing a negative VIRTUAL STATE
- $\gamma = \pi/2$ POLE AT $k = 0$
- $\gamma > \pi/2$ BOUND STATE

STRIKING PHENOMENA IN PARTIAL WAVE AMPLITUDES AT LOW ENERGY: RESONANCES, BOUND & VIRTUAL STATES AND ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE.

The possibility of striking behavior in scattering processes is an important and complex subject. There is much to be learned about the microscopic interactions of particles from the resonances in their scattering cross sections. There is too much material to cover in an orderly fashion — we will discuss a small piece rather carefully as an introduction. You may find more extensive treatments in Gottfried and elsewhere.

Consider scattering from a potential with spherical symmetry. The phase shifts are given by

$$\tan \delta_\ell(k) = \frac{x j_\ell'(x) - \beta_\ell j_\ell(x)}{x n_\ell'(x) - \beta_\ell n_\ell(x)} \qquad (1)$$

where $x = kb$, $k = \sqrt{2mE/\hbar^2}$ and b is a distance greater than the range of the interaction: $V(r) \cong 0$ for $r > b$.

β_ℓ is the log derivative of the radial wavefunction

$$\beta_\ell = \frac{b}{A_\ell(b)} \frac{d}{dr} A_\ell(r) \Big|_{r=b}$$

From eq (1) one obtains the scattering amplitude $f_\ell(k) = \dfrac{1}{k(\cot \delta_\ell - i)}$ and note that this definition of f_ℓ can be analytically continued to complex k, since the Bessel functions are analytic as are the solutions to the Schroedinger equation. Of course the physically interesting region for scattering is k <u>real and positive</u> (by convention)

We study low energies, $|x| \ll 1$

$$\tan \delta_\ell(k) = \frac{j_\ell(x)}{n_\ell(x)} \; \frac{x j_\ell'(x)/j_\ell(x) - \beta_\ell}{x n_\ell'(x)/n_\ell(x) - \beta_\ell}$$

The following low x limits are easily obtained from any book on Bessel's functions:

$$\text{as } x \to 0 \qquad \frac{j_\ell(x)}{n_\ell(x)} \approx -\frac{x^{2\ell+1}}{(2\ell+1)!!\,(2\ell-1)!!}\left(1+O(x^2)\right)$$

$$\frac{x\,j_\ell'(x)}{j_\ell(x)} \approx \ell - \frac{x^2}{2\ell+3} + O(x^4)$$

$$\frac{x\,n_\ell'(x)}{n_\ell(x)} \approx -\ell-1+\frac{x^2}{2\ell-1}+O(x^4)$$

As long as the denominator in eq (1) does not happen to vanish,

$$\tan\delta_\ell \sim x^{2\ell+1}$$

and the scattering is weak — except for $\ell=0$. From here on we treat $\ell=0$ separately and consider what happens if the denominator <u>is</u> small for some $x_0 \ll 1$.

Consider the denominator for $\underline{\ell \neq 0}$:

$$-D(x) \cong \ell+1+\beta_\ell - x^2/(2\ell-1) + O(x^4)$$

and suppose $\underline{\beta_\ell = -\ell-1}$ for some $x_0 = k_0 b$ with x_0 real and $0 < x_0 \ll 1$.

Then $\beta_\ell \cong -\ell-1+(k-k_0)\dfrac{d\beta_\ell}{dk} + O(k-k_0)^2$

and $D(x)=0$ has a solution for $x_1 = x_0 + O(x_0^2)$. Explicitly

$$(x-x_0)\frac{d\beta_\ell}{dx} - \frac{x^2}{2\ell-1} + \dots = 0 \qquad \text{(omitted terms ... are negligible)}$$

$$x_1 = x_0 + \frac{x_0^2}{(2\ell-1)}\left(\frac{d\beta_\ell}{dx}\right)^{-1}_{x_0} + \dots$$

So, to leading order near $x = x_0$ we may write

$$D(x) \cong -(k-k_0)\left.\frac{d\beta_\ell}{dk}\right|_{k_0}$$

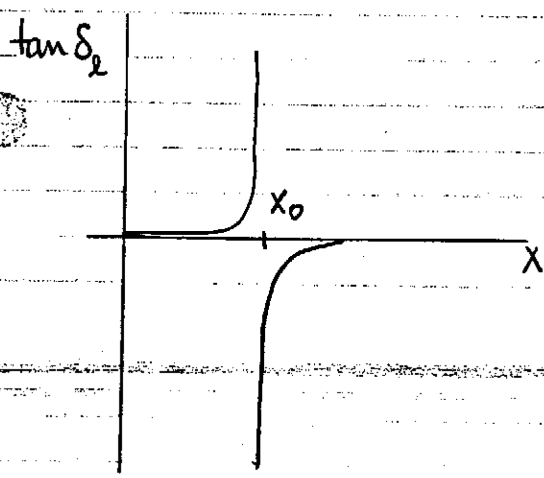
In the rest of the expression for $\tan\delta_\ell$ we do not have to be so careful and can replace each term by its leading form as $x \to 0$. The result is

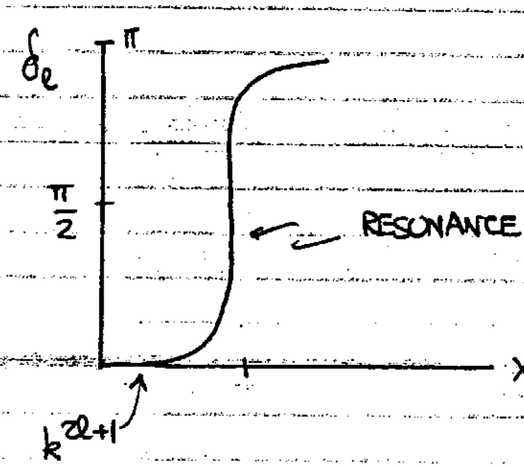
$$\tan\delta_\ell(k) \approx \frac{\Gamma_\ell/2}{k_0 - k}$$

where $$\Gamma_\ell/2 = -\frac{x_0^{2\ell+1}}{[(2\ell-1)!!]^2} \left.\frac{d\beta_\ell}{dk}\right|_{k_0}$$

Soon we will show (but not prove) that $d\beta_\ell/dk$ is negative definite, so Γ_ℓ is >0.

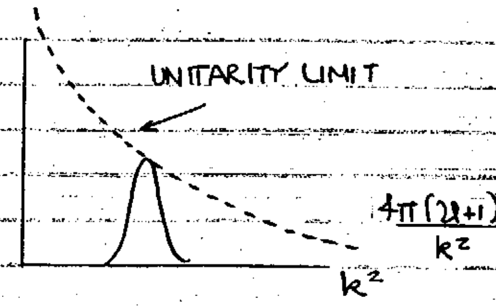
Remember that expect near x_0, $\tan\delta_\ell \approx 0$ so we have





So δ_ℓ grows by π at resonance because $\Gamma_\ell > 0$.

• Behavior of cross section near resonance:



$$\sigma_\ell(k) = \frac{4\pi(2\ell+1)\sin^2\delta_\ell}{k^2}$$

$$= \frac{4\pi(2\ell+1)}{k_0^2}\;\frac{\frac{\Gamma_\ell^2}{4}}{\left[(k_0-k)^2+\Gamma_\ell^2/4\right]}$$

"LORENTZIAN

$\frac{\Gamma_\ell}{2}$ measures width $\quad \underline{\Gamma_\ell = \text{FWHM}}$

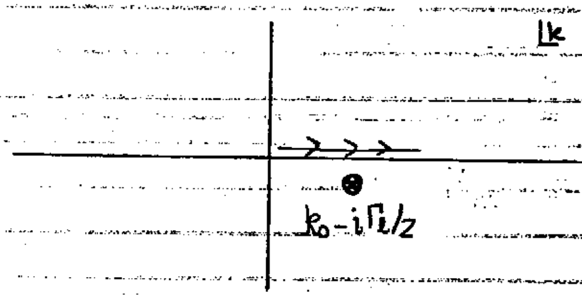
• Behavior of T matrix:

$$\tilde{f}_\ell = kT_\ell = \frac{1}{(\cot\delta - i)} \sim \frac{\Gamma_\ell/2}{(k_0 - k - i\Gamma_\ell/2)}$$

Note $\tilde{f}_\ell = \sin\delta_\ell e^{i\delta_\ell}$ so δ_ℓ is phase of $\tilde{f}_\ell$.

So $\tilde{f}_\ell$ has a pole at $k_p = k_0 - \frac{i\Gamma_\ell}{2}$ provided $\Gamma_\ell/2$ is small so the approximation $k \approx k_0$ is still valid at k_p.

Assuming (to be proved later) γ_ℓ is positive, pole is in lower half plane



as k increases, phase jumps by $+\pi$ going around this simple pole.

The rapid variation in the scattering amplitude can be attributed to the proximity of this pole.