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SUPPLEMENTARY NOTES ON NATURAL UNITS

These notes were prepared by Prof. Jaffe for the 1997 version of Quantum Physics III. They provide background and examples for the material which will be covered in the first lecture of the 2006 version of 8.06.

UNITS

We are all used to the fact that a choice of appropriate units can simplify the appearance of equations. Different units are natural for different problems. Car mechanics like to measure power in *horsepower* , electrical engineers prefer *watts* and particle physicists prefer MeV^2. Each to his/her own... It seems like a pretty dull subject. However, in the realm of modern physics a careful examination of the choice of units leads to some useful insights into the way the universe works. In this section I will first review the *cgs system*. Then I will introduce the system that quantum physicists have named *natural units*. Although it sounds arrogant, these really are the natural units for the quantum world. Next I will describe some consequences of the use of natural units to describe quantum phenomena.

1 The *cgs* System of Units

All of us are familiar with the *cgs* system where *all* physical quantities are expressed in terms of fundamental units of length (ℓ), the centimeter, mass (m), the gram, and time (t), the second . Other quantities that arise in mechanics, like momentum, energy and viscosity have units that are derived from defining equations.[1]. Because $p = mv$, $E = \frac{1}{2}mv^2 + \ldots$, and $dF_x/dA =$

[1]In the subsequent equations $[x]$ is to be read "the dimensions of x."

$\eta \partial v_x / \partial y$ we know that

$$
\begin{aligned}
[\text{momentum}] &= m\ell t^{-1} \\
[\text{energy}] &= m\ell^2 t^{-2} \\
[\text{force}] &= m\ell t^{-2} \\
[\text{viscosity}] &= m\ell^{-1} t^{-1} \quad .
\end{aligned} \tag{1}
$$

Of course practitioners introduce convenient abbreviations: For example, a gm $\text{cm}^{-1}\text{sec}^{-1}$ is a *poise*, a gm $\text{cm}^2\text{sec}^{-2}$ is an *erg* . From a *cgs* point of view, any other unit used in mechanics, like a *foot*, an *atmosphere* or an *acre* merely represents a convenient short hand for so-many $\text{gm}^a\text{cm}^b\text{sec}^c$.

The *cgs* system becomes less intuitive and more confusing to students when we leave the realm of mechanics. Consider, for example, electrodynamics. When a new concept such as *charge* is encountered, it seems necessary to introduce a new unit to measure its quantity. In the case of charge, both the *coulomb* and the *Faraday* were introduced in this way. Now we know that each of them corresponds to the electric charge of some number of electrons. The Coulomb is the magnitude of the charge of approximately $6.24150636 \times 10^{18}$ electrons. The Faraday is the charge of an Avogadro's Number of electrons.

However, the need for an independent unit for electric charge went away when the dynamical laws of electrostatics were worked out. Coulomb's Law enables us to measure charge *using the same units we used in mechanics*, ℓ, m, and t. The reason is that Coulomb's Law tells us the force produced by charges at a fixed distance,

$$F = e_1 e_2 / r^2. \tag{2}$$

Both sides of this equation must have the same dimensions, so

$$[\text{charge}] = [\text{force}]^{1/2}\ell = m^{1/2}\ell^{3/2}t^{-1} \quad . \tag{3}$$

So the basic unit of charge in the *cgs* system is the $\text{gm}^{1/2}\text{cm}^{3/2}\text{sec}^{-1}$(!) The "trick" here was to write Coulomb's Law *without* any constant of proportionality. In eq. (2), when $e_1 = e_2 = 1$ and $r = 1$ cm, then $F = 1$ dyne. Thus the *cgs* unit of charge is that charge which produces a force of 1 dyne at a separation of 1 cm from an equal charge. Since this is a cumbersome notation the unit is given its own name: the esu or stat Coulomb. Nevertheless, the unit of charge is a *derived* quantity in the *cgs* system. Of course it can also be expressed as the charge of so many electrons (approximately 2.0819424×10^9), however it has a fundamental connection to the *cgs* system that the other units of charge do not.

The *MKS* system is different. New, *ad hoc* units are introduced liberally, and proportionality constants are introduced into equations to preserve the meaning of independently defined units. In *MKS* the Coulomb is *defined* to be the charge of so many electrons. To accommodate this *ad hoc* definition a constant $(1/4\pi\epsilon_0)$ must be added to Coulomb's law in *MKS*.

$$F = \frac{1}{4\pi\epsilon_0} Q_1 Q_2 / r^2. \tag{4}$$

The $1/4\pi\epsilon_0$ is necessary in order to specify the force between two one-Coulomb charges separated by one meter.[2]

The argument by which Coulomb's law allows one to measure charge in *cgs* units can be extended to all quantities that arise in electrodynamics. The definition of electric field tells us its units: $\vec{F} = e\vec{E}$, so $\vec{E}$ has dimensions $\mathrm{m}^{1/2}\ell^{-1/2}\mathrm{t}^{-1}$. A further choice and a slight complication arises when magnetism is introduced. Units for the magnetic field can be introduced through the Lorentz Force Law,

$$\vec{F} \propto e\vec{v} \times \vec{B}. \tag{5}$$

However we can change the units for the magnetic field, B, by choosing the constant of proportionality in (5). When electromagnetic radiation is important, it is most convenient to use a system where the electric and magnetic fields are measured in the same units. To accomplish this, the proportionality constant in (5) must have dimensions of *1/velocity*. Electrodynamics offers a natural candidate for this velocity: c — the velocity of light. Thus the Lorentz Force Law reads

$$\vec{F} = e\vec{E} + e\frac{\vec{v}}{c} \times \vec{B} \tag{6}$$

This variant of the *cgs* system is known as the *Gaussian* system of units. It requires introducing a few other factors of c into common electromagnetic

[2]For a more comprehensive discussion of electromagnetic units see the Appendix on units in J. D. Jackson, *Classical Electrodynamics*.

formulas. Here are some example of the dimensions of quantities encountered in electrodynamics in the *cgs* system.

$$\begin{aligned}
[\text{resistivity}] &= t \\
[\text{resistance}] &= \ell^{-1}t \\
[\text{inductance}] &= \ell^{-1}t^2 \\
[\text{magnetic field}] &= m^{1/2}\ell^{-1/2}t^{-1}
\end{aligned} \tag{7}$$

2 Natural Units

The *cgs* and related systems are convenient, practical system for most macroscopic applications. When we leave the scale of human dimensions to study very small sizes and very energetic processes, the *cgs* system is no longer so natural. Centimeters, grams and seconds are not particularly appropriate units for the micro-world. This is reflected in the appearance of large exponents in quantities like the mass of the electron or Planck's constant expressed in *cgs* units. When relativity and quantum mechanics are important, the fundamental constants $\hbar$ (Planck's constant) and c (the speed of light) set natural scales for action and velocity. A system of units in which action and velocity are measured in terms of $\hbar$ and c respectively, has won wide acceptance among atomic, nuclear, particle and astrophysicists, and theorists of all kinds. These *natural units* have the at-first-glance bizarre chracteristic that *all physical quantities are measured in electron volts (to the appropriate power)* . We will use this system almost exclusively for the remainder of Quantum Physics III. The purpose of this section is to provide an introduction to natural units for those who have not seen them before.

It seems self-evident that the units of *mass, length and time* are fundamental and independent. A choice of units for m, and ℓ, for example, does not seem to force us to choose some particular unit for t.[3] As we have just

[3] Actually one could use gravity and the principle of equivalence to define a unit of mass in terms of the units of length and time. Ignoring the distinction between gravitational and inertial mass, Newton's Law (with Newton's constant set equal to unity), $F = m_1 m_2/r^2$, would provide natural units for mass just as Coulomb's Law provides natural units for charge: Define the fundamental unit of mass to be the mass that causes a test mass to accelerate at 1 cm/sec^2 at a distance of 1 cm. In these "gravitostatic units", the unit of mass, the *gsu* in analogy to the *esu* is cm^3/sec^2, and 1 gsu $\approx 1.5 \times 10^7$gm. Such units are

discovered, the example of *cgs* units shows that we do not need to consider the units of any other physical quantity to be fundamental (of course we discussed only mechanics and electrodynamics, but the approach is quite general). I will assume that we need three basic, independent physical quantities on which to build a system of units. There is no deep reason to use mass, length and time as the basic units of the system. One could choose any three independent quantities as the fundamental units in which all physical objects are measured. In the case of natural units the three basic independent units are those of *action*, *velocity* and *energy*. Remembering

$$\begin{aligned} [\text{action}] &= m\ell^2 t^{-1} \\ [\text{velocity}] &= \ell t^{-1} \\ [\text{energy}] &= m\ell^2 t^{-2} \end{aligned} \tag{8}$$

we can express quantities formerly measured in *cgs* in terms of some basic units of action, velocity and energy. Just as we obtain the *cgs* system by choosing the *centimeter*, the *gram* and the *second*; so we obtain the natural system by taking Planck's constant (actually $\hbar = h/2\pi$) as the unit of action, the speed of light (c) as the unit of velocity and the electron volt (eV) as the unit of energy.[4]

$$\begin{aligned} \hbar &\equiv \frac{h}{2\pi} = 1.05457266(63) \times 10^{-27}\,\text{gm cm}^2\text{sec}^{-1} \\ c &= 2.99792458 \times 10^{10}\,\text{cm sec}^{-1} \\ \text{eV} &= 1.60217733(49) \times 10^{-12}\,\text{gm cm}^2\text{sec}^{-2} \quad . \end{aligned} \tag{9}$$

Any quantity D, expressed in *cgs* units can be re-expressed in natural units,

$$[D] = m^a \ell^b t^c = E^\alpha \hbar^\beta c^\gamma, \tag{10}$$

with

$$\begin{aligned} \alpha &= a - b - c \\ \beta &= b + c \\ \gamma &= b - 2a. \end{aligned} \tag{11}$$

"natural" for classical gravity, but are presumably not in common use because we deal mainly with non-gravitational forces.

[4]Actually, the fact that c is quoted without errors in (9) is a signal that it is used as a fundamental unit in the *cgs* system too. The *second* is defined in terms of the frequency of a specific atomic spectral line, and *centimeter* is defined in terms of c and the *second.*

With (10) and (12) we can proceed to express physical quantities in natural units. Here are some examples:

$$\begin{aligned}
[\text{mass}] &= \text{eV}c^{-2} \\
[\text{time}] &= (\text{eV})^{-1}\hbar \\
[\text{length}] &= (\text{eV})^{-1}\hbar c \\
[\text{momentum}] &= \text{eV}c^{-1} \\
[\text{force}] &= (\text{eV})^{2}\hbar^{-1}c^{-1} \\
[\text{pressure}] &= (\text{eV})^{4}\hbar^{-3}c^{-3} \\
[\text{charge}^2] &= \hbar c \\
[\text{magnetic field}] &= (\text{eV})^{2}\hbar^{-3/2}c^{-3/2} \quad .
\end{aligned} \tag{12}$$

3 Advantages of the natural system of units

I suspect that the advantages of this system are not yet apparent. Here are the primary reasons why research physicists prefer it.

3.1 Simplicity

The first great advantage — and the great confusion for non-experts — comes when we *suppress mentioning* the factors of $\hbar$ and c, *leaving all physical quantities measured in units of electron volts*. Such a step could have been taken in the *cgs* system too. We could, for example, suppress the *cm* and *sec* and measure all quantities as some power of a fundamental unit of mass, the *gram.* This is not done for two reasons: first, because there is nothing particularly fundamental about one second or one centimeter so we are not eager to suppress the label which tells us that time was measured in seconds and length in centimeters; and second, because we are used to having a *different* set of units for every different physical quantity — thus, for example, momentum and energy (compare (2) and (13)) have different units in the *cgs* system, but they would both be measured in grams if we suppressed *cm* and *sec*. If you quoted an answer to a calculation in grams, you would have to tell your reader whether it was a momentum or an energy before he would be able to evaluate it in *cgs* units. In the case of natural units the first disadvantage is eliminated: $\hbar$ and c *are* natural units for action and velocity in fundamental physics; and the second disadvantage is outweighed by the great advantage of measuring all quantities in the same units. One must be careful, however, to specify the physical quantity of interest to avoid confusing things measured in the same powers of eV. The problem of converting back from natural units to *cgs* units is made easier by conversion factors

$$\begin{aligned} \hbar c &= 197.327053(59)\,\text{MeV fm} \\ \hbar &= 6.5821220(20) \times 10^{-22}\,\text{MeV sec} \end{aligned} \tag{13}$$

($1\,\text{MeV} = 10^6\,\text{eV}$, $1\ \text{fm} = 10^{-13}\,\text{cm}$). Note $\hbar c$ is equal to unity in natural units.

3.2 Naturalness

$\hbar$ and c set the scale of the quantum world. When we use them as the basis of our unit system we naturally incorporate fundamental properties of the system under study. This is best illustrated by example. Here are several. Please study them carefully — they make the case for the use of natural units in quantum mechanics.

Example 1:

The energy equivalent of the electron's rest mass is 511 keV, so in natural units, the electron mass is

$$m_e = 511\ \text{keV} \quad .$$

What length is $[511\ \text{keV}]^{-1}$?

Answer:

$$\begin{aligned} \ell_e &= \frac{\hbar}{m_e c} = \frac{\hbar c}{m_e c^2} = 197\,\text{MeV fm}/511\ \text{keV} \\ &= 385\,\text{fm} = 3.85 \times 10^{-11}\,\text{cm} \quad . \end{aligned}$$

This is the electron's Compton wavelength.

What time is $[511\ \text{keV}]^{-1}$?

Answer:

$$t_e = \frac{\ell_e}{c} = 1.28 \times 10^{-21}\,\mathrm{sec}$$

which is the time it takes light to travel an electron's Compton wavelength.

What frequency is 511 keV?

Answer:

$$\nu_e = 1/t_e = 7.8 \times 10^{20}\,\mathrm{hz}$$

which is the frequency of each of the two gamma-rays emitted when an electron and positron annihilate.

The moral of this example is that all of the interesting quantum and relativistic scales associated with the electron are naturally encoded in natural units.

Example 2:

An electron with kinetic energy 10 eV scatters at an angle of 0.2 radian from an atom. What length scale structure within the atom does it probe? Answer: First calculate its momentum:

$$\begin{aligned} p &= \sqrt{2mE} \\ &= (2 \times 511\,\mathrm{keV} \times 10\,\mathrm{eV})^{1/2} \\ &= 3.2\,\mathrm{keV} \quad . \end{aligned}$$

For small angles the momentum transfer is approximately $\Delta p = \theta p$. Use the uncertainty principle:

$$\begin{aligned} \Delta p &\cong 0.2p = 0.64\,\mathrm{keV} \ , \\ \Delta x &\cong \hbar/\Delta p = (0.64\,\mathrm{keV})^{-1} \ . \end{aligned}$$

Restore *cgs* units:

$$\Delta x \cong 197\,\mathrm{MeV\ fm}/0.64\,\mathrm{keV} \cong 3.1\,\text{Å}$$

Example 3:

According to (13), e^2 — the square of the electron's charge — has the same units $m\ell^3/t^2$ as the combination $\hbar c$. Thus $e^2/\hbar c$ is a dimensionless measure of the strength of *relativistic, quantum* electrodynamics! What is the value of $e^2/\hbar c$? Answer:

$$\begin{aligned} e &= 4.803 \times 10^{-10}\,\mathrm{esu} \\ e^2 &= 2.307 \times 10^{-19}(\mathrm{esu})^2 \\ (1\,\mathrm{esu})^2 &= 1\,\text{dyne-cm}^2 = 1\,\mathrm{gm\ cm^3/sec^2} \\ \hbar c &= 3.161 \times 10^{-17}\,\mathrm{gm\ cm^3/sec^2} \\ e^2 &= 2.307 \times 10^{-19}/2.161 \times 10^{-17}(\hbar c) = 1/137(\hbar c) \quad . \end{aligned}$$

$e^2/\hbar c \equiv \alpha$ is known as the "fine structure constant." Its precise measured value is $\alpha = (137.0359895(61))^{-1}$.

Example 4:

What is the energy of interaction of the magnetic moment of an electron in the magnetic field of a proton at a distance of 1 Å, when the spins are parallel.

Answer:

$$\begin{aligned} E &= -\vec{\mu}_e \cdot \vec{B} = \frac{\vec{\mu}_e \cdot \vec{\mu}_p}{r^3} = -\frac{\mu_e \mu_p}{r^3} \\ \mu_e &= \frac{e\hbar}{2m_e c} \\ \mu_p &= 2.793\left(\frac{e\hbar}{2m_p c}\right) \\ E &= -2.793\frac{e^2\hbar^2}{4m_p m_e r^3 c^2} \quad . \end{aligned}$$

Now quickly:

$$\begin{aligned} e^2 &\rightarrow \alpha = 1/137 \\ \hbar^2 &\rightarrow 1 \\ c^2 &\rightarrow 1 \\ m_p &\rightarrow 938\,\mathrm{MeV} \\ m_e &\rightarrow 511\,\mathrm{keV} \\ 197\,\mathrm{MeV\ fm} = 1 &\rightarrow 1\mathring{A} = 10^5/197\,\mathrm{MeV} \end{aligned}$$

So

$$\begin{aligned} E &= -2.793 \left(\frac{1}{4}\right)\left(\frac{1}{137}\right)\left(\frac{1}{938}\right)\left(\frac{1}{0.511}\right)\left(\frac{197}{10^5}\right)^3 \text{ MeV} \\ &= 8.13 \times 10^{-8}\,\text{eV} \quad . \end{aligned}$$

4 The strength of quantum electromagnetic forces

In the classical regime electromagnetic forces can be made very strong simply by lumping more and more charge on objects held at a fixed distance. In the world of atoms and particles there are limits. With the help of natural units we can find a fundamental, dimensionless quantity that sets the scale for electromagnetic interaction strengths (not surprisingly, $\alpha = e^2/\hbar c$ is part of the answer). Consider two charged particles, electrons for example, with charge e. We cannot localize a particle within a distance smaller than its Compton wavelength.[5] So the Coulomb interaction energy between the two particles is less or order of $V = e^2/(\hbar/mc)$. The natural energy scale to compare with this is the electron rest mass, $\frac{V}{mc^2} = \frac{e^2}{\hbar c} = \alpha$. Since $\alpha \ll 1$ electromagnetic interactions are a relatively small perturbation on the dynamics of electrons. One might speculate on ways to overcome α and make electromagnetism stronger. One suggestion that frequently arises is to use nuclei with charges Ze, with $Z \sim 100$ possible. In some ways electron-nucleus interactions are parameterized by $Z\alpha$. However the fact that the nucleus is an extended charge distribution with a size much greater than its compton wavelength still prevents electron-nucleus electromagnetic interactions getting too strong. When an electron gets close to a nucleus it sees the individual charged protons rather than the whole nuclear charge.

In contrast the *strong* or *nuclear* interaction energy between two protons separated by their compton wavelength exceeds their rest mass. So the *strong interaction* is indeed strong!

[5] If the momentum uncertainty generated by localization exceeds mc we risk creating particle-antiparticle pairs in the attempt to localize. The restriction $\Delta p < mc$ together with the uncertainty principle yields $\Delta x > \hbar/mc$.

5 The scale of quantum gravity

The strength of gravitational forces is determined by Newton's constant, G_N,

$$\begin{aligned} G_N &= 6.67259(85) \times 10^{-8}\mathrm{cm}^3/\mathrm{gm\ sec}^2 \\ &= 6.70711(86) \times 10^{-39}\mathrm{GeV}^{-2}\hbar c^5 \end{aligned} \tag{14}$$

Notice how small G_N is when expressed in natural units. An immediate consequence of this is that the gravitational force between particles is negligibly small. If we repeat the exercise of the previous section, calculating the gravitational interaction energy of two electrons separated by an electron's compton wavelength, we obtain a number of order 10^{-45}. Clearly gravity is entirely negligible when we study ordinary quantum systems. Another, more provocative way to look at (15) starts with the observation that both $\hbar$ and c appear in the expression for G_N in natural units. Thus we expect this number to emerge in a physical theory that attempts to combine gravity, quantum mechanics and relativity. The search for a consistant relativistic quantum theory of gravity is one of the great unsolved problems of modern physics, so it is quite interesting to learn at what scales it might be important. We can quickly read off the natural energy scale ($E_{Planck} = 1/\sqrt{G_N}$) and length scale ($\ell_{Planck} = \sqrt{G_N}$),

$$\begin{aligned} E_{Planck} &= 1.2 \times 10^{19}\mathrm{GeV} \\ \ell_{Planck} &= 1.7 \times 10^{-33} cm \end{aligned} \tag{15}$$

which are named in honor of Max Planck. Returning to the exercise of the previous section, if we had two extraordinarily heavy elementary particles with masses $M_{Planck} = E_{Planck}/c^2$ separated by a distance equal to their Compton wavelength (which is the extraordinarily short distance ℓ_{Planck}) then and only then would gravity become strong: the gravitational interaction energy of these two hypothetical particles would be comparable to their rest masses.

The Planck length and energy are truly breathtaking numbers. The smallest distance scales probed by our most powerful particle accelerators are now approaching 10^{-17}cm, sixteen orders of magnitude shy of Planck length. The heaviest elementary particle we have discovered (the top quark) has a mass of 175 GeV/c^2, about seventeen orders of magnitude less than the Planck mass. The possibility of probing quantum gravity in the laboratory seems impossibly remote. Undeterred, modern theorists try to leap the intervening orders of magnitude, trying to construct theories of quantum gravity whose natural scales are the Planck length, energy and mass.

6 Natural quantum scales in electrodynamics

Since much of our work this semester will concern quantum phenomena involving electrons and electromagnetic forces, I want to take a closer look at natural quantum scales in electrodynamics.

6.1 Scales in the atom

The natural quantum length scale associated with the electron is its Compton wavelength, $\lambda\!\!\!^{-}_e = \frac{\hbar}{m_e c}$. The dimensionless quantity, $\alpha = \frac{e^2}{\hbar c}$ enables us to fashion new length scales involving different powers of e. For example, multiplying $\lambda\!\!\!^{-}$ by α we get

$$r_e = \frac{e^2}{mc^2} \tag{16}$$

where $\hbar$ has dropped out. r_e is the scale of a *classical* charge distribution whose potential energy is of order the electron's rest mass. It is known as the *classical radius* of the electron and does not involve quantum mechanics at all. Since classical physics breaks down at distances much larger than r_e it should not be surprising that r_e plays almost no role in the quantum structure of electrons, atoms, *etc.*

More interesting is the quantity formed by dividing $\lambda\!\!\!^{-}$ by α,

$$a_0 = \frac{\hbar^2}{me^2}, \tag{17}$$

which is none other than the *Bohr radius* ($\approx \frac{1}{2} \times 10^{-8} cm$) familiar from elementary atomic physics. Notice that a_0 is the only quantity with dimensions of length that can be formed from combinations of e, m and $\hbar$ alone, without c. Therefore a_0 is the only length scale that can characterize non-relativistic quantum effects in the atom. Furthermore the only non-relativistic (*i.e. independent of* c) energy scale for atomic binding is $E_0 = \frac{e^2}{a_0} = \frac{me^4}{\hbar^2}$, which is confirmed by the fact that the binding energy of the ground state of hydrogen is $-\frac{1}{2}E_0$.

We can deduce some rather non-trivial results from this line of reasoning: Since α is dimensionless, $\frac{e^2}{\hbar}$ must have dimensions of velocity. Therefore the *velocity of the electron in the hydrogen atom must be of order* $\frac{e^2}{\hbar}$, or $\frac{v}{c} \approx \alpha \approx \frac{1}{137}$ — seemingly non-trivial result to obtain from dimensional considerations alone.

Finally suppose we are interested in small corrections to the energy levels of the hydrogen atom. Later in the term we will consider the corrections due to the magnetic interactions between the spins of the electron and the proton. We can borrow the result from Example 4 — the interaction energy, ΔE, is of order $\frac{e^2}{r^3}$, where r must be of order the Bohr radius. So $\Delta E \propto e^8 \propto \alpha^4 \propto \alpha^2 E_0$. However, they cannot be considered in isolation because relativistic corrections to E_0 are of the same order. Why? Because they are of order $\frac{v^2}{c^2}E_0$, and $\frac{v}{c} \propto \alpha$. We conclude that any consistant treatment of spin-spin forces in the hydrogen atom must also take into account at least the first relativistic corrections as well.

6.2 The Quanta of Conductance and Flux

Since the units of action and angular momentum are the same, $\hbar$ makes a natural appearance in quantum mechanics as the quantum of angular momentum. We no longer think twice that molecular, atomic, nuclear and particle angular momenta are quantized in multiples of the fundamental unit of angular momentum, $\hbar$. In problems involving electrodynamics we can fashion several other *fundamental quanta* out of the basic constants $\hbar$, c and e, and we can look for dynamical observables that might be quantized as integer multiples of these quanta. Two examples have played an important role in modern quantum physics, *electrical conductance* and *magnetic flux*.

6.2.1 Conductance

A look back at (7) shows that resistance is measured in units of *velocity*$^{-1}$, so *conductance* is measured in units of *velocity*. Conductance should grow with the strength of the electric charge e and should be independent of the sign of e (since the sign of the charge carriers cannot be determined from Ohm's law, $\vec{j} = \sigma \vec{E}$). A quantum of conductance that fills all these requirements is $\frac{e^2}{\hbar}$ — the same as the velocity encountered in the analysis of atoms. So we should expect that for systems where quantum effects dominate the conductance, it comes in multiples of the fundamental *quantum of conductance*, $\sigma_0 = \frac{e^2}{\hbar}$. Later in this course we will encounter two examples of this phenomenon: the Integer Quantum Hall Effect, and Landauer Conductivity in mesoscopic systems. In both cases small systems exhibit quantization of the their conductance in units of σ_0.

There is a suprising feature of this result: in classical electrodynamics conductance is a specimen dependent quantity obtained by multiplying the more fundamental *conductivity* by an effective cross sectional area and dividing by an effective wire-length. One might expect that conductivity would have some fundamental significance, but not the conductance. However, the units of conductivity are [conductivity] $= t^{-1}$, and it is not possible to construct a quantity with those units by combining powers of e, $\hbar$ and c alone. In fact, the quantum phenomena that exhibit quantization of conductance do deal directly with the conductance rather than the conductivity because they involve global aspects of the system (as we shall see). Perhaps this unintuitive feature explains why quantization of conductance waited so long to be discovered.

6.2.2 Flux

Referring back again to (7) we see that the natural units of magnetic flux (Φ) are $[B \times \text{Area}] = m^{1/2}\ell^{3/2}t^{-2}$. These are the same units as the electric charge, e. We would not expect Φ to be quantized in units of e for several reasons: first $\hbar$ does not appear; second, magnetic effects involving particle motion are first order in $\frac{v}{c}$, so we would expect c to appear in the fundamental quantum of flux. Both of these flaws are easily resolved if we consider instead $\Phi_0 = \frac{e}{\alpha} = \frac{\hbar c}{e}$. It may seem surprising that e appears in the denominator here. However, we shall see the flux quantum Φ_0 emerge explicitly in our study of both Landau levels and the Aharonov-Bohm effect.

Perhaps there are other known or yet-to-be-discovered examples of quantization of observables that could be guessed on the basis of dimensional analysis of the kind we have just considered. Perhaps one of you in this course will be the person to work it out!