Quantum Physics III (8.06) Spring 2006
Assignment 10

May 3, 2006 **Due FRIDAY May 12, 2006**

- Please remember to put your name **and section time** at the top of your paper.
- Prof. Liu will give a review session from 2-3:30pm on Friday May 19 in 4-370.
- Additional office hours by Prof. Levitov, Ilya Sigalov and Prof. Liu will soon be announced.
- Your FINAL EXAM is MONDAY MAY 22, 1:30PM-4:30PM, in JOHNSON ICE RINK.

Readings

- Griffiths Chapter 11
- Ohanian Chapter 11
- Prof. Jaffe's notes on scattering

Problem Set 10 and Study Guide

The first four problems are your problem set, due on Friday May 12. The last seven problems will not be graded, but should help you to study for the final exam. Solutions to all problems will be provided.

1. **Scattering from a Reflectionless Potential (10 points)**

   Consider a particle of mass m moving in one dimension under the influence of the potential

   $$V(x) = -\frac{\hbar^2 a^2}{m}\text{sech}^2(ax)\ .$$

   (a) This potential has a normalizable bound state with wave function $\psi_0(x) \propto \text{sech}(ax)$. What is its energy?

   (b) Show that

   $$\psi(x) = \left(\frac{k}{a} + i\tanh(ax)\right)\exp ikx$$

   is a solution to the same problem with energy $E = \hbar^2 k^2/2m$.

(c) Now consider scattering of a particle with energy E from $V(x)$. Explain (should be brief) that the solution of part (b) satisfies the boundary conditions appropriate for this scattering problem, with the particle incident from the left. Use this solution to show that the reflection coefficient is zero, and to determine the transmission coefficient $T(E)$. Show $|T(E)| = 1$.

(d) Show that $T(E)$ has a pole at the energy of the bound state.

2. **Simple Properties of Cross Sections (15 points)**

Scattering in three dimensions introduces some new concepts: cross sections, scattering amplitudes, solid angle, to name a few. This problem should help you understand the basics.

Consider a scattering wave function in three dimensions parametrized by a function $f(\theta, \phi)$:

$$\psi(r, \theta, \phi) = e^{ikz} + \frac{f(\theta, \phi)}{r} e^{ikr} \ .$$

The first term describes an incident plane wave. The second term describes the scattered flux, scattered off some potential localized in the vicinity of $r = 0$. This scattering wave function is only valid at large r. $f(\theta, \phi)$, which parametrizes the scattered flux, is called the scattering amplitude.

The probability flux for the Schrödinger equation is given by

$$\vec{S} = \frac{\hbar}{2mi} \left(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^* \right) \ .$$

(a) Compute the incident flux. Calculate the scattered flux for $\theta \neq 0$. [Note: when calculating the scattered flux, keep only the dominant term at large r.]

(b) Define the cross section per unit solid angle by

$$\frac{d\sigma}{d\Omega} d\Omega = \lim_{r \to \infty} \frac{\vec{S}_{\text{scattered}} \cdot \hat{r}}{|S_{\text{incident}}|} dA \ ,$$

where S_{incident} is the incident flux, $S_{\text{scattered}}$ is the scattered flux, and dA is a small element of area, $dA = r^2 d\Omega$, on a distant sphere.

Show that

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2 \ .$$

[Note that Griffiths denotes $d\sigma/d\Omega$ by the symbol $D(\theta, \phi)$. This notation is unconventional, but it is helpful in reminding one what $d\sigma/d\Omega$ depends on.]

(c) From considerations of flux conservation, derive the optical theorem:

$$\sigma_{\text{elastic}} \equiv \int d\Omega |f(\theta, \phi)|^2 = \frac{4\pi}{k} \text{Im} f(\theta = 0) \ .$$

Hint: Prof. Jaffe's notes are helpful.

3. **Born Approximation for Scattering From Yukawa and Coulomb Potentials, plus a Practical Example of the Latter (15 points)**

Make sure you are aware of Griffiths' Examples 11.5 and 11.6 on page 415 as you do this problem. He has done some of the work for you.

Consider a Yukawa potential

$$V(r) = \beta \frac{\exp(-\mu r)}{r}$$

where β and μ are constants.

(a) Evaluate the scattering amplitude, the differential cross section $d\sigma/d\Omega$, and the total cross section in the first Born approximation. Express your answer for the total cross section as a function of the energy E.

(b) Take $\beta = Q_1 Q_2$ and $\mu = 0$, and show that the differential cross section you obtain for scattering off a Coulomb potential is the same as the classical Rutherford result. Use this differential cross section in part (d) below.

(c) Differential cross sections are what physicists actually use to calculate the rate at which scattered particles will enter their detectors. The number of particles scattered into solid angle $d\Omega$ per second by a single scatterer is given by

$$\frac{d^2 N}{dt d\Omega} = \frac{d\sigma}{d\Omega} \times \frac{d^2 N}{dt dA}$$

where $d^2 N/dtdA$ is the incident flux in units of particles per second per unit area, ie per unit cross sectional area transverse to the beam. Consider a uniform beam of dN/dt particles per second with a cross sectional area A. This beam strikes a target with density n (n is the number of scattering sites per unit volume) and thickness t.

Give an expression for the number of particles scattered into a detector with angular size $d\Omega$ per unit time.

Show that your result is independent of the cross sectional area of the beam even if the beam is not uniform across this area. [Note that this is important, because it is typically easy for an experimenter to measure dN/dt but hard for her to measure either A or the uniformity of the beam across the cross sectional area.]

(d) Consider a beam of alpha particles ($Q_1 = 2e$) with kinetic energy 8 MeV scattering from a gold foil. Suppose that the beam corresponds to a current of 1 nA. [It is conventional to use MKS units for beam currents. 1 nA is 10^{-9} Amperes, meaning 10^{-9} Coulombs of charge per second. Each alpha particle has charge $2e$, where $e = 1.6 \times 10^{-19}$ Coulombs.] Suppose the gold foil is 1 micron thick. You may assume the alpha particles scatter only off nuclei, not off electrons. You may also assume that each alpha particle scatters only once. You will need to look up the density of gold and the nuclear charge of gold (Q_2). How many alpha particles per second do you expect to be scattered into a detector which occupies a cone of angular extent ($d\theta = 10^{-2}$ radians, $d\phi = 10^{-2}$ radians) centered at $\theta = \pi/2$?

(e) Suppose you now move the detector around (keeping it at the same distance from the target and thus keeping the solid angle subtended by the detector the same.) How does the number of particles per second seen in the detector depend on the angular location of the detector, θ? What is the number of particles per second seen in the detector for $\theta = 10°$, $\theta = 45°$ $\theta = 135°$, $\theta = 170°$?

4. **The Size of Nuclei (10 points)**

In lecture we derived an expression for the scattering amplitude in the Born approximation for the elastic scattering of a particle of mass m and charge -$|e|$ from a charge distribution $|e|\rho(\vec{r})$:

$$f(\vec{q}) = \frac{2me^2}{\hbar^2 q^2} \int d^3 r e^{-i\vec{q}\cdot\vec{r}} \rho(\vec{r}) \ .$$

Recall that $\vec{q} = \vec{k}' - \vec{k}$ is the momentum transferred to the scattered particle in the collision. For elastic scattering, $q \equiv |\vec{q}| = 2|\vec{k}| \sin(\theta/2)$. If the electrons used in a scattering experiment are relativistic, $k \simeq E/c$.

(a) The charge distribution of a nucleus is *not* localized at a mathematical point. f is therefore not exactly that for Rutherford scattering. The charge distribution is roughly constant out to a radius R and then drops rapidly to zero. A simple model is:

$$\rho(\vec{r}) = \frac{3Z}{4\pi R^3} \quad \text{for } r \leq R$$

and $\rho = 0$ for $r > R$. Calculate the cross section for electron scattering from such a nucleus as a function of q^2.

(b) The ratio of the actual amplitude for scattering from a point nucleus is called the "form factor". Sketch the form factor as a function of qR.

The form factor tells us about the "shape" of the charge distribution in a nucleus, and thus tells us how the protons within a nucleus are arranged. In our simple model, the form factor tells us the value of R. If nuclei had precisely the shape we have used in our simple model, experimenters would measure a form factor with precisely the functional form you have calculated, and would then do a fit to obtain a measurement of R, the radius of the nucleus.

(c) For relativistic electrons with energy E, if you are able to count the scattered electrons at a variety of angles, ranging from θ close to zero to θ close to π, what range of q can you access? If you use electrons with $E \ll 1/R$, show that you will not be able to make an accurate determination of R. You will not be able to "resolve" the fact that scattering off a nucleus differs from Rutherford scattering.

The values of R for nuclei are around $(2-7)\times 10^{-13}$ cm. Roughly how large an electron energy do you need in order to do a reasonable measurement of R?

First aside: The above problem uses a simple model, but it is not all that far from the real thing.

Second aside: The next step in the process of unveiling the structure of matter on smaller and smaller length scales was the discovery that the protons and neutrons that make up a nucleus have substructure. Electron beams with energies appropriate for studying nuclear structure (ie the distribution of protons within a nucleus, which you've been analyzing in this problem) cannot resolve the substructure of a proton. Thus, the discovery of the quark structure of the proton had to wait until the construction of the SLAC linear accelerator, which began accelerating electrons to 21 GeV in the late 1960's. In 1967, Jerome Friedman, Henry Kendall and Richard Taylor began the series of experiments in which quarks were discovered. When an 21 GeV electron scatters at large angles off a quark in a proton, the proton does not remain intact. This means that the description of these experiments requires an understanding of *inelastic* scattering. In an inelastic collision, the scattered electron's momentum changes by $\vec{q}$, and its energy also changes.

THE PROBLEMS BELOW SHOULD NOT BE HANDED IN. THEY WILL NOT BE GRADED. THEY ARE INTENDED AS A STUDY GUIDE TO HELP YOU UNDERSTAND SCATTERING THEORY AND THE GROVER ALGORITHM. SOLUTIONS WILL BE PROVIDED.

5. **The Born Approximation in One Dimension (15 points)**

   (a) Do Griffiths Problem 11.16.

   [You need not derive the Green's function as Griffiths does in his text. Rather, it is sufficient for you to take the answer Griffiths gives, and show that it is indeed the integral equation for the one dimensional Schrödinger equation.]

   (b) Do Griffiths Problem 11.17.

   (c) Do Griffiths Problem 11.18.

6. **Scattering from a Small Crystal (8 points)**

   We want to investigate the structure of a crystal by scattering particles from it. The particle sees the potential

$$V(\vec{x}) = \sum_i v(\vec{x} - \vec{X}_i)$$

   where the $\vec{X}_i$ are the position vectors of the scattering atoms and $v(\vec{x})$ is the scattering potential of a single atom. Assume that v is weak enough that we can use the Born approximation for the whole crystal, ie for V.

   (a) Express the differential cross section as the product of two factors, one of which depends on v and the other on the structure of the crystal, ie the set of points $\vec{X}_i$. Both factors will depend on the momentum transfer $\vec{q}$.

   (b) Briefly, compare to whatever you know about Bragg scattering.

7. **Partial Waves (10 points)**

   Suppose the scattering amplitude for a certain reaction is given by

$$f(\theta) = \frac{1}{k}\left(\frac{\Gamma k}{k_0 - k - ik\Gamma} + 3e^{2i\beta k^3}\sin 2\beta k^3 \cos\theta\right) \tag{1}$$

   where Γ, k_0, and β are constants characteristic of the potential which produces the scattering. Of course $k = \sqrt{2mE/\hbar^2}$ is the deBroglie wavenumber.

   - What partial waves are active (*i.e.* what values of ℓ)?

- What are the phase shifts in the active partial waves? Do they have the proper behavior as $k \to 0$?
- What is the differential cross section, $d\sigma/d\Omega$ for general values of k?
- What are the partial wave cross sections, σ_ℓ?
- Assume $\beta k_0^3 \ll 1$. Give an approximation to the total cross section $\sigma(k)$ for $k \approx k_0$.
- What is the total cross section for general values of k? What is the imaginary part of the forward scattering amplitude? Do they satisfy the optical theorem?

8. **Combining Born and Partial Waves (6 points)**

   A potential $V(r)$ is of the Yukawa form,

   $$V_{\text{Yukawa}}(r) = \beta \frac{\exp(-\mu r)}{r}$$

   for $r > R$, but is unknown for $r < R$. The differential cross-section $d\sigma/d\Omega$ has been measured as a function of energy $(= \hbar^2 k^2/2m)$ and angle, for values of k up to and of the order of $1/R$. Attempts to fit $d\sigma/d\Omega$ to the partial wave formulae using a small number of phase shifts $\delta_\ell(k)$ (i.e. putting $\delta_\ell = 0$ for all ℓ's greater than some ℓ_0) have been a miserable failure. [Aside: this is what actually happened when the scattering of neutrons off protons was first done at energies up to about 100 MeV. The fits to a straightforward partial wave analysis of the type just described were hopelessly ambiguous.]

   It is proposed that though the higher phase shifts *are* all small, their sum cannot be neglected. This is annoying, because we do not want to use the partial wave analysis for infinitely many values of ℓ. In order to make progress, we make the following assumptions:

   - Because of the centrifugal potential $\hbar^2\ell(\ell+1)/2mr^2$, once ℓ is "big enough" the behavior of $V(r)$ for $r < R$ does not matter, and we can therefore calculate the behavior of δ_ℓ for large ℓ just from $V_{\text{Yukawa}}(r)$.
   - These δ_ℓ can be expanded in powers of β and we can keep only the first term. That is, each of these δ_ℓ's is small, so we can expand it. It is the sum of all of them which is causing problems.

   Carry out the following procedure to implement this idea, treating $\ell \geq 1$ as "big enough" in the sense above.

   (a) Write the Born approximation for the scattering amplitude $f_{\text{Yukawa}}(\theta, \phi)$ for the potential V_{Yukawa}. You can get this from Problem Set 9, so this part of the problem is worth no points.

(b) Calculate the mean value of f_{Yukawa} at fixed k, averaging over all directions (θ, ϕ), and subtract this mean value from f_{Yukawa}.

[Since the s-wave scattering amplitude has nonzero mean, while all higher partial wave scattering amplitudes have zero mean, what you have just done is to subtract the s-wave part of the Born approximation to f_{Yukawa}. What remains is the sum of the contributions of all partial waves with $\ell \geq 1$. Note that making the first Born approximation is equivalent to linearizing f_{Yukawa} in β. So, you have now accomplished most of what we set out to do.]

(c) You must now add back in an expression for the s-wave contribution to f. As the s-wave is more sensitive to the small r region of the potential than any other partial wave, you cannot analyze it using the Yukawa potential. So, just add in the s-wave contribution to f written in terms of an unknown phase shift $\delta_0(k)$.

Write down a formula expressing $d\sigma/d\Omega$ in terms of m, $\hbar$, k, θ, μ, β and $\delta_0(k)$.

[To complete the story, what you now do is fit the data to your formula, and thus obtain μ, β and $\delta_0(k)$. The fit to the data now works beautifully, and the fitted value of μ turns out to be the mass of the pion (times $c/\hbar$) just as Yukawa had predicted 20 years before.]

9. **Scattering from a δ-Shell (13 points)**

Consider s-wave ($\ell = 0$) scattering from the potential

$$V(r) = \lambda \frac{\hbar^2}{2mR} \delta(r - R)$$

with λ a large positive constant. To find the phase shift $\delta_0(k)$ we have to solve

$$\frac{d^2 u}{dr^2} + k^2 u = \frac{\lambda}{R} \delta(r - R) u \ ,$$

with $u = 0$ at $r = 0$ and $u = \sin(kr + \delta)$ for $r > R$.

(a) What is u in $r < R$?

(b) By comparing $u'(r)/u(r)$ just inside and just outside $r = R$, find a formula to determine δ.

(c) Find the scattering length a, defined by $\lim_{k \to 0} \delta_0 = -ka$.

(d) Assume $\lambda \gg 1$. Sketch $\delta(k)$. Show that for kR just below $n\pi$, with n a positive integer, $\delta(k)$ increases very rapidly by π (as kR increases towards $n\pi$). Sketch the s-wave cross-section σ_0. Show that the s-wave scattering from this potential is the same as that from a hard sphere of radius R for all values of kR *except* those such that kR is close to $n\pi$. What is the significance of these values?

10. **Ramsauer-Townsend Effect (6 points)**

At very low energies only the s-wave contributes to scattering. If, for some reason, the s-wave phase shift vanishes, then so does the scattering amplitude. Under these circumstances a projectile can pass through material without any scattering. This effect is known as the Ramsauer-Townsend Effect.

Consider a three dimensional "square well",

$$V(r) = \begin{array}{ccc} -V_0 & \text{for} & r \leq a \\ 0 & \text{for} & r > a \end{array} \tag{2}$$

(a) Find the condition on $\gamma^2 = 2mV_0a^2/\hbar^2$ such that the cross section for a particle of mass m is zero at zero energy. Your answer should be in the form of a set of values of γ^2, specified graphically. (You need not obtain numerical values, but make sure that your graph is drawn accurately enough and labelled so that someone can use your graph to read off the first few numerical values to within 10%.)

(b) As you can see from part (a), it is useful to think of the Ramsauer-Townsend effect as a function of the depth of the potential. The existence of bound states is also a function of the depth of the potential. Show that if a square well which displays an exact Ramsauer-Townsend effect is made a little deeper or shallower (you have to figure out which) it then has a bound state at threshold.

11. **Scattering in the Semiclassical Approximation (4 points)**

The semiclassical approximation becomes better at high energies. For most problems high energies means scattering as opposed to bound states. It is quite straightforward to estimate the phase shift in the semiclassical approximation.

Consider scattering in the s-wave in three dimensions. The radial wavefunction obeys

$$-u''(k,r) + \frac{2m}{\hbar^2}V(r)u(k,r) = k^2u(k,r) \tag{3}$$

and $u(k,0) = 0$.

In this problem we will assume that $V(r)$ is smooth and slowly varying and that $r^2V(r) \to 0$ as $r \to \infty$ and that $V(r)$ is negative at all r. (We'll change the last assumption in the next problem.)

Recall from 8.05 (and show for yourself if you like) that as $r \to \infty$, $u(k,r) \sim \sin(kr + \delta_0(k))$, where $\delta_0(k)$ is the phase shift.

Show that in the semiclassical approximation

$$\delta_0(k) = \int_0^\infty dr \left[\sqrt{k^2 - \frac{2m}{\hbar^2}V(r)} - k\right] \tag{4}$$

12. **A Semiclassical Analysis of Resonant Scattering (13 points)**

Consider s-wave scattering for a particle of mass m off a potential $V(r)$ which vanishes at the origin, rises steadily as r increases from zero, reaches a maximum at $r = c$, and then goes quickly to zero as r increases further.

For $\ell = 0$, the radial wave function $u(r)$ satisfies the same Schrödinger equation as that for a particle in one dimension with potential V, subject to the boundary condition $u(0) = 0$.

Consider scattering with energy E where $0 \ll E \ll V(c)$. The classical turning points are at $r = a$ and $r = b$ with $a < c < b$.

(a) What is the semiclassical approximation to the wave function in the classically allowed region, $0 \leq r < a$?

(b) What is the ratio of the amplitude of the wave function $u(r)$ in the semiclassical approximation in the region $x > b$ compared to that in the region $x < a$, for generic values of E?

(c) For some special values of E, there is a qualitative change in the ratio of the amplitude for $x > b$ to the amplitude for $x < a$, compared to its "generic" value at other energies. What condition determines these special values of E?

(d) Describe the qualitative behavior of the s-wave phase shift and s-wave cross-section for energies in the vicinity of the special values of E.

13. **The Grover Algorithm**

Consider the 8 dimensional Hilbert space formed by taking the tensor product of the Hilbert spaces for three spin-one-half particles.

We denote the basis states as follows:

$$\begin{aligned}
|0\rangle &= |0,0,0\rangle \\
|1\rangle &= |0,0,1\rangle \\
|2\rangle &= |0,1,0\rangle \\
|3\rangle &= |0,1,1\rangle \\
|4\rangle &= |1,0,0\rangle \\
|5\rangle &= |1,0,1\rangle \\
|6\rangle &= |1,1,0\rangle \\
|7\rangle &= |1,1,1\rangle
\end{aligned}$$

where, for example, $|0,1,0\rangle$ means a state in which all three spins are in eigenstates of S_z, with eigenvalues $+\hbar/2$, $-\hbar/2$, $+\hbar/2$.

Throughout this problem you will be constructing a variety of 8×8 matrices, working in a basis with basis vectors ordered as above.

(a) The first stage of the Grover algorithm is initialization. Suppose we start with all spins up, namely in state $|0\rangle$. We want to find a unitary operator $U_{\text{initialize}}$ such that

$$U_{\text{initialize}}|0\rangle = |s\rangle$$

where the state $|s\rangle$ is given by

$$|s\rangle = \frac{1}{\sqrt{8}}\Bigg[|0\rangle + |1\rangle + |2\rangle + |3\rangle + |4\rangle + |5\rangle + |6\rangle + |7\rangle\Bigg] .$$

Construct the 8×8 matrix $U_{\text{initialize}}$ as the product of three 8×8 unitary matrices each of which acts only within the Hilbert space of one of the three spins.

Note: my guess is that this is the part of this problem that you will find trickiest. Note that you need not do this part of the problem in order to do any of the other parts.

(b) Lets suppose that $f(3) = 1$ and $f(a) = 0$ for $a = 0, 1, 2, 4, 5, 6, 7$. In other words, "3 is the winner". Define a diagonal unitary matrix called $(-1)^f$ that acts on basis states as follows:

$$\begin{aligned} (-1)^f|a\rangle &= |a\rangle \text{ for } a \neq 3 \\ (-1)^f|3\rangle &= -|3\rangle \ . \end{aligned}$$

Write $(-1)^f$ as an 8×8 matrix.

Note: this part of the problem is very easy as posed. Too easy, in fact. Doing it this way is a little too much like "looking inside the black box and seeing how f works". What you should really do is construct this unitary operator by introducing a "work-bit", introducing an operator U_f which represents a function call via $U_f|a, 0\rangle = |a, f(a)\rangle$ and $U_f|a, 1\rangle = |a, 1 - f(a)\rangle$, introducing the operator L defined in lecture, and then constructing $(-1)^f = U_f L U_f$. I do recommend that you do this explicitly, but adding the work bit means doubling the Hilbert space to 16×16 so I am not going to ask you to turn this in.

(c) Write the unitary operator $U_s \equiv 2|s\rangle\langle s| - 1$ as an 8×8 matrix. (You should check that your matrix is unitary, but do not turn this check in.)

(d) Find the state

$$\Bigg[U_s\,(-1)^f\Bigg]^k |s\rangle$$

for $k = 0, 1, 2, 3$. You should find that for $k = 2$, it is fairly close to the state $|3\rangle$ while for $k = 3$, it has become less close to $|3\rangle$.

Suppose that the state with $k = 2$ "is measured", meaning that S_z is measured for each of the three spins. What is the probability that the outcome of this measurement will be $+\hbar$, $-\hbar$, $-\hbar$ (which corresponds to the state $|3\rangle$)? That is, what is the probability that upon measurement you get the right answer?

Note: I proved in lecture that for large N, the best choice for k is the integer closest to $\pi\sqrt{N}/4$. For our $N = 8$, which is not even very large, $\pi\sqrt{N}/4 = 2.22$. Now that you have understood the $N = 8$ example explicitly, you should review the proof of the large-N result. [Note: although it is not really necessary, it is fine if you choose to use a program like Mathematica to multiply out matrices.]