Quantum Physics III (8.06) Spring 2006
Assignment 2

Feb 14, 2006 **Due Wednesday Feb 22, 2006, 7pm**

- Please remember to put **your name and section time** at the top of your paper.

Readings

The readings assigned last week should suffice for most of this problem set.
The reading assignment for the next two weeks is:

- Supplementary notes on Canonical Quantization and Application to a Charged Particle in a Magnetic Field.
- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry's phase for now. We will come back to this later.
- Quite remarkably, given its length, Cohen-Tannoudji never mentions the Aharonov-Bohm effect. It does have a nice treatment of Landau levels, however, in Ch. VI Complement E
- Those of you reading Sakurai should read pp. 130-139.

Problem Set 2

1. **Eigenstates of the "Translation" Operator in tight-banding model (5 points)**

   Consider an infinite dimensional Hilbert space with orthonormal basis states that we will call $|n\rangle$ where n is an integer running from $-\infty$ to $+\infty$.

   The Hamiltonian for the system is

   $$H = \sum_{n=-\infty}^{n=+\infty} \left[E_0 |n\rangle\langle n| \; - \; \Delta |n\rangle\langle n+1| \; - \; \Delta |n+1\rangle\langle n| \right] .$$

   This is the tight-banding model discussed in lecture. Define the "translation" operator T by:

   $$T|n\rangle = |n+1\rangle$$

   You should check for yourself that T commutes H.

   Find the state which is an eigenstate of T with eigenvalue $\exp(-i\theta)$. (Call this state $|\theta\rangle$). Find the energy of $|\theta\rangle$.

2. **Relativistic degenerate electron gas (5 points)**

   Consider a 3-dimensional gas of N free ultra-relativistic (i.e. one can ignore the rest mass of electrons) electrons at zero temperature. Show that the total energy of the system can be written as

   $$E = \frac{3}{4} N \epsilon_F$$

   where ϵ_F is the energy of a single particle state at the Fermi surface.

3. **White dwarfs, Neutron stars and Black holes (10 points)**

   (a) Consider a white dwarf star of the same mass as the Sun. Assume that the star is mainly made of Carbon. What is the radius of star? Note that the Sun has a mass of $M_{sun} = 2 \times 10^{33} g$ and a radius of $R_{sun} = 7 \times 10^5 km$. Find the ratio of the mass density of the white dwarf and the Sun.

   (b) In a neutron star, the neutron degeneracy pressure stabilizes the collapse. Calculate the radius of a neutron star with the mass of the Sun. You can assume that the star only consists of neutrons and the neutron gas is free[1]. Find the (neutron) Fermi energy and compare it to the rest energy of a neutron.

   (c) If a star has a mass M that is larger than the Chandrasekhar mass for a neutron star, the degenerate neutron pressure cannot balance the attractive force of the gravity and the star will collapse to form a black hole. A black hole has a "surface of no return", i.e. any object lying within a radius r_s from the center of gravity of the black hole can not escape and will be devoured by the hole. r_s is called the "Schwarzschild radius". Estimate r_s by combining M with G_N and c to obtain a length. Evaluate your r_s for $M = M_{sun}$.

   (d) Compare various radii you obtained for (a) (b) (c).

4. **The Dirac comb (10 points)**

   The qualitative behavior of solids is dictated to a large extent simply by the fact that the electrons feel a periodic potential. The example we discussed in lecture is called the "tight binding model." The other classic example is the Dirac comb, which Griffiths treats on pages 226-228. You should read through Griffiths' treatment.

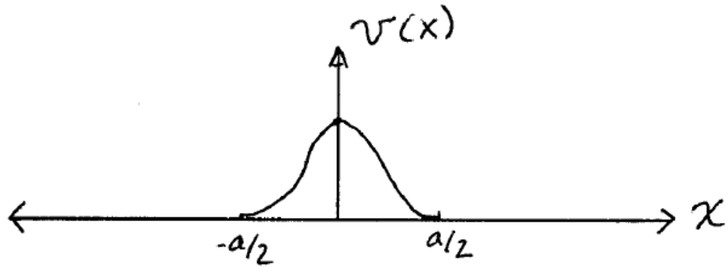
   Do Griffiths problem 5.20.

[1] This is not a very good approximation to the realistic situation in which nuclear interactions between neutrons are important.

5. **Analysis of a general one-dimensional periodic potential (30 points)**

Consider a one-dimensional periodic potential $U(x)$ that we shall choose to view as the sum of lots of identical potential barriers $v(x)$ of width a, centered at the points $x = na$, where n is an integer.

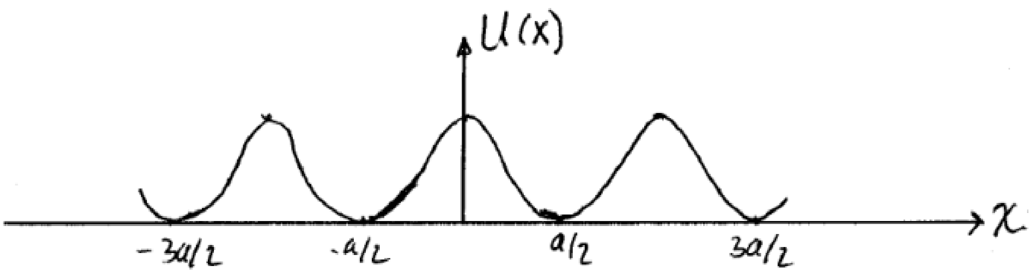
We shall require v to be even, that is $v(x) = v(-x)$, but other than that we shall allow the shape of the barrier to be arbitrary. $v(x) = 0$ for $|x| \geq a/2$. In pictures, $v(x)$ looks like:



The periodic potential is then given by

$$U(x) = \sum_{n=-\infty}^{\infty} v(x - na)$$

and looks like:



Before we analyze U, let us analyze v. For any energy $E > 0$, there are two linearly independent solutions to the Schrödinger equation with the single barrier potential $v(x)$. One, which we shall call $\psi_L(x)$ describes a plane wave incident from the left:

$$\begin{aligned} \psi_L(x) &= \exp(ikx) + r\exp(-ikx) \ , \quad x \leq -a/2 \\ &= t\exp(ikx) \ , \qquad x \geq a/2 \ , \end{aligned} \tag{1}$$

where k is related to E by $E = \hbar^2 k^2/2m$. We shall not need the form of ψ where the potential is nonzero. The other solution with the same energy describes a wave incident from the right:

$$\begin{aligned}\psi_R(x) &= t\exp(-ikx)\ , \qquad x \le -a/2 \\ &= \exp(-ikx) + r\exp(ikx)\ , \quad x \ge a/2\ ,\end{aligned} \tag{2}$$

with the same reflection coefficient r and transmission coefficient t as in (1) because $v(x)$ is even.

We can write the complex number t in terms of its magnitude and phase as

$$t = |t|\exp(i\delta)\ , \tag{3}$$

where δ is a real number known as the phase shift since it specifies the phase of the transmitted wave relative to the incident one. Conservation of probability requires that

$$|t|^2 + |r|^2 = 1. \tag{4}$$

To this point, we have reviewed 8.04 material and established notation.

(a) Let ψ_1 and ψ_2 be any two solutions of the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_i}{dx^2} + v(x)\psi_i = E\psi_i$$

with the same energy. Define the "Wronskian" of these two solutions by

$$W(\psi_1,\psi_2) = \psi_2(x)\frac{d}{dx}\psi_1(x) - \psi_1(x)\frac{d}{dx}\psi_2(x)\ .$$

Prove that W is independent of x by showing that $dW/dx = 0$.

(b) By evaluating $W(\psi_L,\psi_R^*)$, prove that rt^* is pure imaginary, so r must have the form

$$r = \pm i|r|\exp(i\delta) \tag{5}$$

where δ is the same as in (3).

(c) Now, we begin our analysis of solutions of the Schrödinger equation in the periodic potential U. Since $U = v$ in the region $-a/2 \le x \le a/2$, in that region any solution to the Schrödinger equation with potential U must take the form

$$\psi(x) = A\psi_L(x) + B\psi_R(x)\ , \quad -a/2 \le x \le a/2\ , \tag{6}$$

with ψ_L and ψ_R given by (1) and (2). Bloch's theorem tells us that

$$\psi(x+a) = \exp(iKa)\psi(x)$$

and, with $\psi' \equiv d\psi/dx$,

$$\psi'(x+a) = \exp(iKa)\psi'(x) \ .$$

By imposing these conditions at $x = -a/2$, show that the energy of the electron is related to K by

$$\cos Ka = \frac{t^2 - r^2}{2t} \exp(ika) + \frac{1}{2t} \exp(-ika) \tag{7}$$

with k specifying the energy via

$$E = \hbar^2 k^2 / 2m \ .$$

[Note that some of you may succeed in deriving an expression relating all the quantities in (7) — and no other quantities — but then not succeed in reducing your expression to the form (7). If so, you will not lose many points. And, make sure to use (7), rather than whatever you obtain, in the following parts.]

(d) Show that as a consequence of (4), (5) and (7) the energy and K of the Bloch electron are related by

$$\cos Ka = \frac{\cos(ka + \delta)}{|t|} \ . \tag{8}$$

Note that $|t|$ is always less than one, and becomes closer and closer to one for larger and larger k because at high incident energies, the barrier becomes increasingly less effective. Because $|t| < 1$, at values of k in the neighborhood of those satisfying $ka + \delta = n\pi$, with n an integer, the right hand side of (8) is greater than one, and no solution can be found. The regions of E corresponding to these regions of k are the energy gaps.

(e) Suppose the barrier is very strong, so that $|t| \approx 0$, $|r| \approx 1$. Show that the allowed bands of energies are then very narrow, with widths of order $|t|$. [Note: this is the tight-binding case, discussed in lecture. This is the case that applies to a deeply bound atomic energy level which in a crystal becomes a narrow band. In this case, because the energy level is well below the top of the barrier between single-atom potential wells, "transmission" requires tunnelling, meaning that $|t|$ is small.]

(f) Suppose the barrier is very weak (so that $|t| \approx 1$, $|r| \approx 0$, $\delta \approx 0$). Show that the energy gaps are then very narrow, the width of the gap containing $k = n\pi/a$ being $2\pi n \hbar^2 |r| / ma^2$. [Note: this shows that the continuum states – namely those whose energies are above the top of the barriers – are also separated into bands. The gaps between the bands get narrower and narrower for higher and higher energy continuum states.]

(g) Show that in the special case where $v(x) = +\alpha\delta(x)$ where $\delta(x)$ is the Dirac delta function — i.e. the Dirac comb model discussed in Griffiths p.226-228 — the phase shift and transmission coefficent are given by

$$\cot\delta = -\frac{\hbar^2 k}{m\alpha}$$

and

$$|t| = \cos\delta$$

and that (8) becomes the expression derived in Griffiths.