Quantum Physics III (8.06) Spring 2006
Assignment 3

Feb 17, 2006 **Due Feb 28, 2006, 7pm**

- Please remember to put **your name and section time** at the top of your paper.

Readings

The current reading assignment is:

- Supplementary notes on Canonical Quantization and Application to a Charged Particle in a Magnetic Field.
- Griffiths Section 10.2.4 is an excellent treatment of the Aharonov-Bohm effect, but ignore the connection to Berry's phase for now. We will come back to this later.
- Cohen-Tannoudji Ch. VI Complement E
- Those of you reading Sakurai should read pp. 130-139.

Problem Set 3

1. **Classical motion in a Magnetic Field (10 points)**

   Consider a particle of mass m and charge q moving along a trajectory $\vec{x}(t)$ through an electric field $\vec{E}$ and magnetic field $\vec{B}$, which are specified by scalar and vector potentials $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ via

   $$\vec{E} = -\vec{\nabla}\phi(\vec{x},t) - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}(\vec{x},t) \ . \tag{1}$$

   As discussed in lecture, the classical Hamiltonian for this system is

   $$\mathcal{H}(\vec{x},\vec{p}) = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \ . \tag{2}$$

   where $\vec{p}$ is the canonical momentum.

(a) (optional, bonus credit: 5 points) Show that the Hamilton equations derived from this Hamiltonian are

$$m\vec{v} = \vec{p} - \frac{q}{c}\vec{A}, \qquad \vec{v} = \dot{\vec{x}} \tag{3}$$

and the force law

$$m\ddot{\vec{x}} = \frac{q}{c}\vec{v} \times \vec{B} + q\vec{E} \ . \tag{4}$$

[Note that equation (3) implies that canonical momentum $\vec{p}$ is NOT the same as kinetic momentum $m\vec{v}$. This is due to that we have a *velocity-dependent* force.]

(b) (5 points) Now set $\vec{E} = 0$ and consider a constant magnetic field along z-direction, i.e. $B_x = B_y = 0$, $B_z = B =$ const. Classically in the $x - y$ plane the particle travels in a circle around a "center of orbit" with an angular velocity ω_L given by

$$\omega_L = \frac{qB}{mc} \ . \tag{5}$$

Suppose that the "center of orbit" has coordinates (x_0, y_0). Show that x_0, y_0 can be expressed in terms of the coordinates (x, y) and the velocities (v_x, v_y) of the particle as

$$x_0 = x + \frac{v_y}{\omega_L}, \qquad y_0 = y - \frac{v_x}{\omega_L} \ . \tag{6}$$

2. **Gauge Invariance and the Schrödinger Equation (14 points)**

In canonical quantization of (2) one promotes $\vec{x}$ and $\vec{p}$ into operators with canonical commutation relations (we use hat to denote quantum operators)

$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk} \ , \quad \text{i.e.} \quad \hat{p}_i = -i\hbar\frac{\partial}{\partial x_j} \ . \tag{7}$$

Motivated by classical equation (3), we introduce velocity operators, defined by

$$\hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{q}{c}A_i\right) \tag{8}$$

The quantum Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2m}\left(\hat{\vec{p}} - \frac{q}{c}\vec{A}(\vec{x}, t)\right)^2 + q\phi(\vec{x}, t) \\ &= \frac{1}{2}m\left(\hat{v}_x^2 + \hat{v}_y^2 + \hat{v}_z^2\right) + q\phi(\vec{x}, t) \end{aligned} \tag{9}$$

with the corresponding time dependent Schrödinger equation given by

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{q}{c}\vec{A}(\vec{x}, t)\right)^2 \psi(\vec{x}, t) + q\phi(\vec{x}, t)\psi(\vec{x}, t) \ . \tag{10}$$

(a) Consider

$$\vec{A}'(\vec{x},t) = \vec{A}(\vec{x},t) - \vec{\nabla} f(\vec{x},t)$$
$$\phi'(\vec{x},t) = \phi(\vec{x},t) + \frac{1}{c}\frac{\partial f}{\partial t}(\vec{x},t) \ . \tag{11}$$

$(\vec{A}', \phi')$ and $(\vec{A}, \phi)$ describe the same $\vec{E}$ and $\vec{B}$. Show that if $\psi(\vec{x},t)$ solves the Schrödinger equation with $\vec{A}, \phi$ (which we will call "unprimed gauge"), then

$$\psi'(\vec{x},t) \equiv \exp\left(-\frac{iq}{\hbar c} f(\vec{x},t)\right)\psi(\vec{x},t) \tag{12}$$

solves the Schrödinger equation with $\vec{A}', \phi'$ (which we will call "primed gauge").

(b) Show that $\langle\psi|\psi\rangle$ and $\langle\psi|\hat{x}_i|\psi\rangle$ are the same in the primed and unprimed gauges. This means that the identity operator and the operator $\hat{x}_i$ are "gauge invariant operators".

(c) Show that $\langle\psi|\hat{p}_i|\psi\rangle$ is *not* gauge invariant, whereas

$$\langle\psi|\hat{v}_i|\psi\rangle \tag{13}$$

is gauge invariant. Now assuming that f is time-independent, show that the Hamiltonian is a gauge invariant operator[1].

[Conclusion: the "canonical momentum" $\hat{p}_i$ is not a gauge invariant operator, but the "kinetic momentum" $m\hat{v}_i$ is a gauge invariant operator.]

(d) Suppose that $\psi_n(\vec{x})$ is an eigenstate of the Hamiltonian in the unprimed gauge, with eigenvalue E_n. Assume that the gauge transformation function f is time-independent. Show that $\psi'_n(\vec{x})$ is an eigenstate of the Hamiltonian in the primed gauge, with the *same* eigenvalue E_n.

[You have just showed that the spectrum of energy levels, and the degeneracy of each level, are the same in all gauges.]

In 8.05, we said that "physical observables are matrix elements of hermitian operators." We should have said: "physical observables are matrix elements of *gauge invariant* hermitian operators."

[1] When f is time-dependent, $\langle\psi|H|\psi\rangle$ and $\langle\psi'|H'|\psi'\rangle$ are no longer the same. But one can show that differences in expectation values of H are gauge independent, i.e. for any states $|\psi\rangle$ and $|\phi\rangle$ satisfying the Schrodinger equation, $\langle\psi|H|\psi\rangle - \langle\phi|H|\phi\rangle$ is gauge invariant. This weaker form of gauge invariance is enough since only energy differences are observable.

3. **General aspects of quantum motion in a magnetic field (15 points)**

The quantum motion for a particle in a magnetic field shows some resemblances to the classical motion and also many important differences. The differences can be traced to various commutators derived in this problem, in particular equations (14) and (17).

The questions in this problem should be derived without explicitly choosing a gauge.

(a) In this part we consider an arbitrary magnetic field (not necessarily constant). Find the commutators

$$[\hat{v}_i, \hat{v}_j] = ? \tag{14}$$

where $\hat{v}_i$ is the velocity operator defined by

$$\hat{v}_i = \frac{1}{m}\left(\hat{p}_i - \frac{q}{c}A_i\right) \tag{15}$$

What can you conclude about the motion of the particle from (14)?

(b) In this and all parts below, we take $\vec{E} = 0$ and

$$B_x = B_y = 0, \qquad B_z = B = \text{const}$$

and look at the motion in $x - y$ plane only. Motivated by the classical expressions (6) we introduce quantum operators

$$\hat{x}_0 = \hat{x} + \frac{\hat{v}_y}{\omega_L}, \qquad \hat{y}_0 = \hat{y} - \frac{\hat{v}_x}{\omega_L} \tag{16}$$

Find the commutators

$$[\hat{x}_0, \hat{y}_0] = ? \tag{17}$$

You should find (17) non-vanishing. What can you say about the motion in $x - y$ plane from (17)?

(c) Find the commutators

$$[\hat{x}_0, \hat{v}_{x,y}] = ?, \qquad [\hat{y}_0, \hat{v}_{x,y}] = ? \tag{18}$$

(d) Using (18) to show that

$$[\hat{x}_0, H] = [\hat{y}_0, H] = 0 \tag{19}$$

Equations (17) and (19) imply that one of $\hat{x}_0$ and $\hat{y}_0$ (or an arbitrary linear combination of them, but not both) can be diagonalized together with the Hamiltonian.

4. **Electromagnetic Current Density in Quantum Mechanics (12 points)**

The probability flux in the Schrödinger equation can be identified as the electromagnetic current density, provided the proper attention is paid to the effects of the vector potential. This current density will play a role in our discussion of the quantum Hall effect.

Way back in the 8.04 you derived the probability flux in quantum mechanics:

$$\vec{S}(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\vec{\nabla}\psi\right] \ . \tag{20}$$

In the presence of electric and magnetic fields, the probability current is modified to

$$S_i(\vec{x},t) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\partial_i\psi\right] - \frac{q}{mc}\psi^*\psi A_i = \mathrm{Re}\left(\psi^*\hat{v}_i\psi\right) \tag{21}$$

This probability flux is conserved and when multiplied by q, the particle's charge, it can be interpreted as the electromagnetic current density, $\vec{j} \equiv q\vec{S}$.

(a) Derive the expression eq. (21) for the probability flux. [Hint: Choose to work in a gauge where $\vec{\nabla}\cdot\vec{A} = 0$. The derivation of eq. (21) is parallel to that of (20), i.e. you need to derive

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{S}$$

with $\rho = \psi^*\psi$ and $\vec{S}$ given by eq. (21).]

(b) Assuming that ψ has units $1/l^{3/2}$ as one would expect from the normalization condition, $\int d^3x\psi^*\psi = 1$, show that $\vec{j} = q\vec{S}$ has units of charge per unit area per unit time, which are the dimensions of current density.

(c) In part (a), you assumed that $\vec{\nabla}\cdot\vec{A} = 0$. Now show that $\vec{S}$ has *exactly the same form* in any gauge, *ie.* show that $\vec{S}$ is gauge invariant. That is, show that if we make the following transformations, then $\vec{S}'$ defined in terms of $\vec{A}'$ and ψ' is identical to $\vec{S}$ defined in terms of $\vec{A}$ and ψ.

$$\begin{aligned}\vec{A}'(\vec{x},t) &= \vec{A}(\vec{x},t) - \vec{\nabla}f(\vec{x},t)\\ \psi'(\vec{x},t) &= \exp\left(-\frac{iq}{\hbar c}f(\vec{x},t)\right)\psi(\vec{x},t)\end{aligned}$$

where f is any function of $\vec{x}$ and t.

5. **Translation Invariance in a Uniform Magnetic Field (14 points)**

One of the surprising things in our analysis of the quantum mechanics of a particle in a uniform magnetic field is that even though $\vec{B}$ is uniform, and we would therefore expect translation invariance in the xy-plane, we find that, in any gauge we choose, the Hamiltonian does not appear to reflect this symmetry.

The resolution to this question is that translation operators which do commute with the Hamiltonian *can* be constructed. We shall see, however, that there is a catch.

Consider a magnetic field $\vec{B} = (0, 0, B)$. We introduce operators

$$Q_x = -\frac{qB}{c}\hat{y}_0, \qquad Q_y = \frac{qB}{c}\hat{x}_0 \tag{22}$$

and

$$U_a = e^{\frac{i}{\hbar}Q_x a}, \qquad V_b = e^{\frac{i}{\hbar}Q_y b} \tag{23}$$

where $\hat{x}_0, \hat{y}_0$ were defined in (16) and a, b are arbitrary real constants. Now we will show that U_a and V_b are in fact the desired translation operators in x and y directions. Note that due to equations (19), U_a and V_b clearly commute with the Hamiltonian. We work in the gauge in which $\vec{A} = (-By, 0, 0)$. The time-independent Schrödinger equation (for states in the xy-plane) is

$$\frac{-\hbar^2}{2m}\left[\frac{\partial^2\psi}{\partial y^2} + \left(\frac{\partial}{\partial x} + \frac{iqB}{\hbar c}y\right)^2\psi\right] = E\psi \tag{24}$$

(a) Find the explicit expression of Q_x and Q_y by plugging $\vec{A} = (-By, 0, 0)$ into (22).

(b) The appearance of y in (24) destroys (on the face of it) invariance under translation in the y direction. Show, however, that if $\psi(x, y)$ is a solution of (24), then so too is $\tilde{\psi}(x, y)$ defined by

$$\tilde{\psi}(x, y) = \psi(x, y + b)\exp(iqBbx/\hbar c)\ . \tag{25}$$

Translation in x direction is trivial since (24) does not explicitly depend on x.

[Hint: be careful with your notation. Express $(\partial/\partial x + iqBy/\hbar c)\tilde{\psi}$ at the point (x, y) in terms of ψ and $\partial\psi/\partial x$ at the point $(x, y + b)$.]

(c) Show that

$$V_b|\psi\rangle = |\tilde{\psi}\rangle\ . \tag{26}$$

and

$$U_a\psi(x, y) = \psi(x + a, y) \tag{27}$$

Thus U_a and V_b can indeed be interpreted as translation operators.

(d) Calculate

$$[U_a, V_b] =? \tag{28}$$

and show that U_a commutes with V_b if and only if ab is an integer multiple of $A_B = 2\pi\hbar c/qB$ and hence if and only if abB is an integer multiple of $\Phi_0 = hc/q$.

[Note: The intuition behind using $Q_x \propto \hat{y}_0$ and $Q_y \propto \hat{x}_0$ to generate translations in x and y directions is as follows. On the one hand, $\hat{x}_0$ and $\hat{y}_0$ can be interpreted as the operators corresponding to the "center of orbit". On the other hand, as you have showed in equation (17), $\hat{x}_0$ and $\hat{y}_0$ do not commute with each other and thus cannot be diagonalized simultaneously. In fact, equation (17) indicates that they should be considered as a canonical pair with a commutation relation similar to that of a position operator and momentum operator. So they can be considered as the operators which generate translations of each other. That is, $\hat{x}_0$ generates translations in the center of motion in y-direction and $\hat{y}_0$ generates translations in the center of motion in x-direction.]