Quantum Physics III (8.06) Spring 2006
Assignment 4

Feb 22, 2005 **Due March 7, 2005, 6pm**

- Please remember to put your name **and section time** at the top of your paper.

Readings

The reading assignment for this week was already given. You should focus on completing it.

If you wish to read ahead, the next reading assigment will be:

- Griffiths Sections 6.1, 6.2
- Cohen-Tannoudji Chapter XI

Problem Set 4

Remarks on Notations:

1. In this problem set, the following definitions are used throughout:

$$\omega_L \equiv \frac{qB}{mc}, \qquad l_0 \equiv \sqrt{\frac{\hbar}{m\omega_L}} = \sqrt{\frac{\hbar c}{qB}} \tag{1}$$

$$v_x = \frac{1}{m}\left(p_x - \frac{q}{c}A_x\right), \qquad v_y = \frac{1}{m}\left(p_y - \frac{q}{c}A_y\right) \tag{2}$$

2. In this problem set, we will no longer put hat on quantum operators as in pset 3.

3. To avoid confusion with their eigenvalues, the operator $\hat{x}_0, \hat{y}_0$ discussed in pset 3 will now be called X and Y, which are defined by

$$X = x + \frac{v_y}{\omega_L}, \qquad Y = y - \frac{v_x}{\omega_L} \tag{3}$$

1. **Landau Levels: numerics (4 points)**

   Consider a charged particle of charge q and mass m in a constant magnetic field B. The energy eigenvalues are separated by a spacing $\hbar\omega_L$ and typical wave functions are characterized by the length l_0 (see equation (1) for notations).

Suppose B is a field of 10 Tesla and $q = e$ (electron charge). (This is a very strong magnetic field but is certainly one which can be created in the laboratory.) In a 10 Tesla magnetic field, what is $\hbar\omega_L$ in eV? What is ℓ_0 in cm?

Useful facts: 1 Tesla $= 10^4$ gauss. The gauss is the cgs unit of B. This turns out to mean that if B is 1 gauss, then the force eB is 300 eV/cm. Also, $\hbar c = 197 \times 10^{-7}$eV cm. And, the mass of the electron is $m = 0.511\text{MeV}/c^2$.

2. **Transformation between basis vectors of different gauges (12 points)**

Consider a particle of charge q and mass m moving in a constant magnetic field $\vec{B} = (0, 0, B)$. In the gauge

$$A_x = -By, \qquad A_y = A_z = 0 \tag{4}$$

we found in lecture that a basis of energy eigenvectors in the lowest Landau level is given by (see equation (1) for notations)

$$\psi_0(x, y; y_0) = e^{-\frac{i}{l_0^2} y_0 x} \phi_0(y - y_0) \tag{5}$$

where

$$\phi_0(y - y_0) = \frac{1}{(\pi l_0^2)^{\frac{1}{4}}} e^{-\frac{(y-y_0)^2}{2l_0^2}} \tag{6}$$

is the ground state wave function for a harmonic oscillator of frequency ω_L and

$$y_0 = -\frac{c\hbar k_x}{qB} = -l_0^2 k_x \ .$$

The state vector (5) is a common eigenvector of H and $Y = y - \frac{1}{\omega_L} v_x$ with eigenvalues $\frac{1}{2}\hbar\omega_L$ and y_0 respectively.

In the gauge

$$A'_y = Bx, \qquad A'_x = A'_z = 0 \tag{7}$$

completely parallel discussion to that in lecture leads to the basis of energy eigenvectors in the lowest Landau level

$$\psi'_0(x, y; x_0) = e^{\frac{i}{l_0^2} x_0 y} \phi_0(x - x_0) \tag{8}$$

with

$$x_0 = l_0^2 k_y \ .$$

The state vector (8) is a common eigenvector of H and $X = x + \frac{1}{\omega_L} v_y$ with eigenvalues $\frac{1}{2}\hbar\omega_L$ and x_0 respectively.

You will now find the transformation between (5) and (8).

(a) Show that equations (7) and (4) are related by

$$\vec{A}' = \vec{A} - \nabla f$$

for some function $f(x, y)$ and find f.

(b) In pset 3 you have shown that (note there X, Y were called $\hat{x}_0, \hat{y}_0$ respectively)

$$[X, Y] = -il_0^2 \tag{9}$$

Denote $|x_0\rangle$ the eigenvector of X with eigenvalue x_0 and $|y_0\rangle$ the eigenvector of Y with eigenvalue y_0. From (9) show that

$$\langle x_0| = \int_{-\infty}^{\infty} \frac{dy_0}{\sqrt{2\pi l_0^2}}\, e^{-\frac{ix_0y_0}{l_0^2}}\, \langle y_0| \ .$$

There is no need to derive the normalization factor $\frac{1}{\sqrt{2\pi l_0^2}}$ in above equation, which is chosen by convention.

(c) Check explicitly that (8) can be written in terms of linear superpositions of (5) as

$$\psi_0'(x, y; x_0) = e^{-\frac{iq}{\hbar c}f} \int_{-\infty}^{\infty} \frac{dy_0}{\sqrt{2\pi l_0^2}}\, e^{\frac{ix_0y_0}{l_0^2}}\, \psi_0(x, y; y_0) \tag{10}$$

where f is what you find from (a). The factor $e^{-\frac{iq}{\hbar c}f}$ in (10) comes from the gauge transformation. [Hint: To prove (10), it is convenient to write equation (6) in terms of its Fourier transform

$$\phi_0(y - y_0) = \frac{l_0}{(\pi l_0^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{ik(y-y_0) - \frac{l_0^2k^2}{2}}$$

and use it in the right hand side of (10).]

3. **Landau levels in symmetric gauge (21 points)**

In this problem we discuss wave functions for a particle of charge q and mass m in a constant magnetic field $\vec{B} = (0, 0, B)$ in the symmetric gauge

$$\vec{A} = \left(-\frac{1}{2}By, \frac{1}{2}Bx, 0\right) \ . \tag{11}$$

(a) In this part we first re-derive the spectrum of the Hamiltonian without assuming any gauge. In pset 3, you have shown that the velocity operators defined in (2) satisfy the commutation relation (without taking any explicit gauge)

$$[v_x, v_y] = \frac{i\hbar\omega_L}{m} \ . \tag{12}$$

Introduce the annihilation and creation operator

$$a = \sqrt{\frac{m}{2\hbar\omega_L}}(v_x + iv_y), \qquad a^\dagger = \sqrt{\frac{m}{2\hbar\omega_L}}(v_x - iv_y) \ . \tag{13}$$

Show that

$$[a, a^\dagger] = 1$$

and the Hamiltonian can be written as

$$H = \hbar\omega_L(a^\dagger a + \frac{1}{2}) \tag{14}$$

The above shows that the spectrum of the theory is given by that of a simple harmonic oscillator with frequency ω_L.

(b) Now introduce complex coordinates

$$z = x + iy, \qquad \bar{z} = x - iy \tag{15}$$

and

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \ . \tag{16}$$

Show that in gauge (11), a defined in (13) can be expressed in terms of complex coordinates as

$$a = -\frac{i}{\sqrt{2}}\left(\frac{z}{2l_0} + 2l_0\partial_{\bar{z}}\right) \tag{17}$$

where l_0 was defined in (1).

(c) Show that wave functions of the lowest Landau level, which satisfy

$$a|\psi\rangle = 0 \tag{18}$$

can be written as

$$\psi(z, \bar{z}) = \exp\left(-\frac{z\bar{z}}{4l_0^2}\right) f(z) \tag{19}$$

with $f(z)$ an arbitrary holomorphic function (i.e. f is a function of z only and does not depend on $\bar{z}$).

(d) From (19), a basis of normalized energy eigenstates for the lowest Landau level can be chosen to be

$$\psi_n = N_n z^n \exp\left(-\frac{z\bar{z}}{4l_0^2}\right), \qquad n = 0, 1, \cdots \tag{20}$$

Find the normalization constant N_n and show

$$\langle\psi_n|x^2 + y^2|\psi_n\rangle = 2(n+1)l_0^2 \tag{21}$$

(e) Plot the contourplot of the wave function (20) in the $x - y$ plane for $n = 0, 4, 10$ (take $l_0 = 1$) using Mathematica (or other similar software).

[Note: For n not too small the probability density of (20) is a ring (of width l_0) round the origin. Equation (21) tells you roughly how radius of the ring changes with n.]

(f) Introduce the canonical angular momentum operator

$$L = xp_y - yp_x \tag{22}$$

Show that in the symmetric gauge (11)

$$[L, H] = 0$$

Express L in terms of complex coordinate $z, \bar{z}$ and show that

$$L\psi_n = n\hbar\psi_n$$

Thus ψ_n is the basis in which L and H are simultaneously diagonalized.

4. **Counting States in the lowest Landau Level in the symmetric gauge (6 points)**

In lecture we counted the number of states in the lowest Landau level in the Landau gauge for a material of finite size. In this problem we do the counting in the symmetric gauge.

(a) Find the value of r_{max} that the probability density

$$dP = 2\pi r|\psi_n|^2(r)$$

achieves maximum, where ψ_n is given by (20).

(b) Now consider a material of the shape of a disk with radius R. We will assume that the radius R is much larger than l_0 which is the characteristic scale of the wave functions (in problem 1, you got a sense of the magnitude of l_0) and ignore the boundary effects.

The condition $r_{max} < R$ then gives the largest value n_{max} allowed by the size of the material. Show that

$$n_{max} \approx \frac{\Phi}{\Phi_0}$$

where Φ is the total magnetic flux through the sample and $\Phi_0 = \frac{hc}{e}$ is the basic flux quantum.

5. **Coherent state in the symmetric gauge (12 points)**

The classical Larmor motion is most closely approximated by a coherent state, just like for the harmonic oscillators. In this problem we construct these coherent states using symmetric gauge (11).

(a) Find the eigenstate ψ_0 of the annihilation operator (17) with eigenvalue $-i\frac{z_0}{\sqrt{2}l_0}$, i.e.

$$a\,\psi_0 = -i\frac{z_0}{\sqrt{2}l_0}\psi_0 \tag{23}$$

You do not need to normalize ψ_0.

(b) Calculate the probability density $|\psi_0|^2$. Show that it is a Gaussian localized at $(x, y) = (x_0, y_0)$ for $z_0 = x_0 + iy_0$.

(c) Suppose $\psi(t = 0) = \psi_0$, find $\psi(t)$ at time t. (Hint: Remind yourself the analogous problem for a harmonic oscillator.)

(d) Calculate the probability density $|\psi(t)|^2$. Show that it a Gaussian centered at $z(t) = z_0 e^{-i\omega_L t}$.

[Note: $z(t)$ can be interpreted as the orbit of a classical particle circling around the origin. Using the translation operator discussed in p5 of pset 3 one can translate the origin of the motion to an arbitrary location.]

6. **Off-Diagonal Conductance in Two-Dimensions (10 points)**

Consider a two-dimensional strip of material of length L and width W, placed in a magnetic field perpendicular to the strip and with an electric field established in the direction of the length L.

(a) Suppose that the resistivity matrix is given by the classical result

$$\rho = \begin{pmatrix} \rho_0 & -\rho_H \\ \rho_H & \rho_0 \end{pmatrix} \tag{24}$$

where $\rho_H = B/nec$ is the Hall resistivity and ρ_0 is the usual Ohmic resistivity. Find the conductivity matrix, $\sigma = \rho^{-1}$. Write it in the form:

$$\sigma = \begin{pmatrix} \sigma_0 & \sigma_H \\ -\sigma_H & \sigma_0 \end{pmatrix} . \tag{25}$$

What are σ_0 and σ_H?

(b) Suppose $B = 0$, so the Hall resistivity is zero. Notice that the Ohmic conductivity, σ_0, is just $1/\rho_0$. In particular, note that $\sigma_0 \to \infty$ as $\rho_0 \to 0$. Now suppose $\rho_H \neq 0$. Show that $\sigma_0 \to 0$ as $\rho_0 \to 0$, so it is possible to have *both* σ_0 and ρ_0 equal to zero

(c) Now consider the case where $\sigma_0 = \rho_0 = 0$. Show that the Hall *resistance* of the sample is equal to the Hall resistivity, and is therefore independent of L and W. (You should begin by showing that in two dimensions, unlike in three dimensions, resistance has the same dimensions as resistivity. This alone is not enough to argue that the resistance and resistivity are equal. Dimensional analysis alone does not preclude the occurence of factors like L/W in the resistance. Indeed, if there were "on-diagonal" contributions to σ, you would find that the resistances do depend on L and W.)