Quantum Physics III (8.06) Spring 2006
Assignment 5

March 3, 2006 **Due Tuesday March 14, 2006**

- Please remember to put your name **and section time** at the top of your paper.
- Remember that your **midterm** will be on **Thursday March 23**, in class. (i.e. 12:30pm to 2:00 pm in 4-370.)

Readings

The reading assignment for this problem set and part of the next one is:

- Griffiths all of Chapter 6.
- Cohen-Tannoudji Chapter XI including Complements A-D.

Problem Set 5

1. **The Aharonov-Bohm Effect on Energy Eigenvalues (20 points)**

   You have seen the "standard presentation" of the Aharonov-Bohm effect in lecture. The standard presentation has its advantages, and in particular is more general than the presentation you will work through in this problem. However, students often come away from the standard presentation of the Aharonov-Bohm effect thinking that the only way to detect this effect is to do an interference experiment. This is not true, and the purpose of this problem is to disabuse you of this misimpression before you form it.

   As Griffiths explains on pages 385-387 (344-345 in 1st Ed.), the Aharonov-Bohm effect modifies the energy eigenvalues of suitably chosen quantum mechanical systems. In this problem, I ask you to work through the same physical example that Griffiths uses, but in a different fashion which makes more use of gauge invariance.

   Imagine a particle constrained to move on a circle of radius b (a bead on a wire ring, if you like.) Along the axis of the circle runs a solenoid of radius $a < b$, carrying a magnetic field $\vec{B} = (0, 0, B_0)$. The field inside the solenoid is uniform and the field outside the solenoid is zero. The setup is depicted in Griffiths' Fig. 10.10. (10.12 in 1st Ed.)

   (a) Construct a vector potential $\vec{A}$ which describes the magnetic field (both inside and outside the solenoid) and which has the form $A_r = A_z = 0$ and $A_\phi = \alpha(r)$ for some function $\alpha(r)$. We are using cylindrical coordinates z, r, ϕ.

(b) Now consider the motion of a "bead on a ring": write the Schrödinger equation for the particle constrained to move on the circle $r = b$, using the $\vec{A}$ you found in (a). Hint: the answer is given in Griffiths.

(c) Solve the Schrodinger equation of (b) and find the energy eigenvalues and eigenstates.

(d) In this and the following parts we find the energy eigenvalues using a different gauge.

Since $\vec{\nabla} \times \vec{A} = 0$ for $r > a$, it must be possible to write $\vec{A} = \vec{\nabla} f$ in any *simply connected* region in $r > a$. [This is a theorem in vector calculus.] Show that if we find such an f in the region

$$r > a \text{ and } -\pi + \epsilon < \phi < \pi - \epsilon \ ,$$

then

$$f(r, \pi - \epsilon) - f(r, -\pi + \epsilon) \to \Phi \text{ as } \epsilon \to 0 \ .$$

Here, the total magnetic flux is $\Phi = \pi a^2 B_0$. Now find an explicit form for f, which is a function only of ϕ.

(e) Use the $f(\phi)$ found in (d) to gauge transform the Schrödinger equation for $\psi(\phi)$ within the angular region $-\pi + \epsilon < \phi < \pi - \epsilon$ to a Schrödinger equation for a *free* particle within this angular region. Call the original wave function $\psi(\phi)$ and the gauge-transformed wave function $\psi'(\phi)$.

(f) The original wave function ψ must be single-valued for all ϕ, in particular at $\phi = \pi$. That is, $\psi(\pi - \epsilon) - \psi(-\pi + \epsilon) \to 0$ and $\frac{\partial \psi}{\partial \phi}(\pi - \epsilon) - \frac{\partial \psi}{\partial \phi}(-\pi + \epsilon) \to 0$ as $\epsilon \to 0$. What does this say about the gauge-transformed wave function? I.e., how must $\psi'(\pi - \epsilon)$ and $\psi'(-\pi + \epsilon)$ be related as $\epsilon \to 0$?

[Hint: because the $f(\phi)$ is not single valued at $\phi = \pi$, the gauge transformed wave function $\psi'(\phi)$ is not single valued there either.]

(g) Solve the Schrödinger equation for ψ', and find energy eigenstates which satisfy the boundary conditions you derived in (e). What are the allowed energy eigenvalues?

(h) Plot the energy eigenvalues as a function of the enclosed flux, Φ. Show that the energy eigenvalues are periodic functions of Φ with period Φ_0, where you must determine Φ_0. For what values of Φ does the enclosed magnetic field have no effect on the spectrum of a particle on a ring? Show that the Aharonov-Bohm effect can only be used to determine the fractional part of Φ/Φ_0.

[Note: you have shown that even though the bead on a ring is everywhere in a region in which $\vec{B} = 0$, the presence of a nonzero $\vec{A}$ affects the energy eigenvalue spectrum. However, the effect on the energy eigenvalues is determined by Φ, and is therefore gauge invariant. To confirm the gauge invariance of your result, you can compare your answer for the energy eigenvalues to Griffiths' result, obtained using a different gauge.]

2. **Fractional Quantum Hall Effects (5 points)**

We will not be able to study the fractional quantum hall effect in 8.06. However, I want to give you at least a some sense of it, at a qualitative level. At the very least, I want to convey how the discovery of this effect was even more of a surprise than the discovery of the integer quantum hall effect. This problem is a bona fide part of this problem set, and "counts" in that sense. However, the material alluded to in this problem will not appear on the midterm or final.

The fractional quantum hall effect was discovered in 1982 by Tsui, Störmer and Gossard. They studied a very clean sample of the same sort in which the integer quantum hall effect had been discovered two years earlier. At a low temperature, and in a high enough magnetic field, they discovered a plateau with

$$\frac{1}{R_{\rm hall}} = \frac{1}{3}\frac{e^2}{h} \qquad \text{and} \qquad R_{\rm long} = 0 \tag{1}$$

occurring over a range of filling fractions centered at

$$\frac{nhc}{eB} = \frac{1}{3}\ .$$

If we attempt to describe this with 8.06 physics, we would say we have a one-third filled Landau level, and as such would have very many degenerate ground states to choose between, corresponding to the choice of which third of the states that make up the lowest Landau level we fill, and which two thirds we leave empty. Depending on what choice among the many possibilities we made, we could find many different values of $R_{\rm hall}$ and would generically find $R_{\rm long} \neq 0$. This cannot describe the experimental result.

The correct description of the fractional quantum hall effect relies crucially on the Coulomb interaction between the electrons. The state described by the "Laughlin wave function", named after Robert Laughlin who proposed it as a way to understand the experimental results, cannot be described by first solving for single electron wave functions, and then filling some of these while leaving others empty. In other words, it is not a direct product of single electron wave functions. It is an intrinsically many-electron wave function, requiring methods beyond those we shall learn in 8.06 to study. However, we can understand at a qualitative level why its discovery required a very clean sample: in order for the fractional quantum hall effect to be observed, the effects of Coulomb repulsion between the electrons must dominate over the interaction between electrons and impurities. In the time since Tsui et al's discovery, as ever cleaner samples have been studied, plateaus have been seen at more and more fractional hall conductivities, for example at 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, ... and 2/3, 3/5, 4/7, 5/9, 6/11, 7/13, ... and many more. It turns out that all these states are described by nondegenerate ground state wave functions, very different from the massively degenerate states we would construct à la 8.06, starting with single electron states, ignoring Coulomb interactions, and simply filling a fraction of a Landau level.

The purpose of this problem is for you to work out one very surprising consequence of the existence of the "1/3-plateau". Take the experimental facts (1) at $nhc/eB = 1/3$ as given. Imagine inserting an infinitesimally thin solenoid *within* this sample, perpendicular to the plane of the sample. (The flux tube "pokes through" the sample. This is a thought experiment.) Slowly increase the flux through this solenoid from zero to Φ_0, the quantum of flux. Now, consider a circular path, within the sample, encircling the flux tube at a great distance. Use the experimentally measured hall conductance to determine how much charge crosses this circular path as the flux in the solenoid increases from 0 to Φ_0. Call your answer "Q". Determine the magnitude of Q but do not worry about its sign.

You've now completed the problem I've posed for you, but lets take a minute to understand the significance of the answer. From our study of the Aharonov-Bohm effect, we can conclude that the Hamiltonian of the system is the same at the beginning and the end of the shenanigans you have performed above. Flux 0 and flux Φ_0 in your flux tube give the same Hamiltonian for the electrons in the sample. However, because your flux tube penetrates through the sample, during the time the flux was increasing it can have had a nontrivial effect (beyond just the Aharonov-Bohm effect) on those electrons whose wave functions are nonzero at the location of the solenoid, and which thus feel the magnetic field in the solenoid. What this means is that although the Hamiltonian at the end of your shenanigans is the same as at the beginning, the state of the system may now be in an excited state, with a different energy. Assuming that the system started in its ground state, it could now be in an excited state. In fact, this is what has happened. By increasing the flux in the solenoid as you did, you have created an excitation of the system, with some finite but nonzero energy, localized in the vicinity of your flux tube. The calculation you did above determines the charge Q of this excitation. This argument, due originally to Laughlin, shows that (regardless of what the detailed description of its ground state wave function turns out to be) the observed 1/3 quantum hall plateau must be described by a Hamiltonian which includes localized excitations whose charge is Q. This should convince you that its ground state wave function is quite nontrivial! In addition to being nondegenerate, the ground state is also "incompressible". If you think carefully about the thought experiment you analyzed, you should get a sense of the meaning of this term in this context.

3. **Perturbation of the Three-Dimensional Harmonic Oscillator (18 points)**

The spectrum of the three-dimensional harmonic oscillator has a high degree of degeneracy. In this problem, we see how the addition of a perturbation to the Hamiltonian reduces the degeneracy. This problem is posed in such a way that you can work through it before we even begin to discuss degenerate perturbation theory in lecture.

Consider a quantum system described by the Hamiltonian

$$H = H_0 + H_1 \tag{2}$$

where

$$H_0 = \frac{1}{2m}\vec{p}^2 + \frac{1}{2}m\omega^2\vec{x}^2 \tag{3}$$

where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{p} = (p_1, p_2, p_3)$. The perturbing Hamiltonian H_1 is given by

$$H_1 = KL_2 \tag{4}$$

where K is a constant and where $L_2 = x_3p_1 - x_1p_3$.

In parts (a)-(e) of this problem, we study the effects of this perturbation within the degenerate subspace of states which have energy $E = (5/2)\hbar\omega$ when $K = 0$.

(a) Set $K = 0$. Thus, in this part of the problem $H = H_0$. Define creation and annihilation operators for "oscillator quanta" in the 1, 2 and 3 directions. Define number operators N_1, N_2, N_3. Denote eigenstates of these number operators by their eigenvalues, as $|n_1, n_2, n_3\rangle$. What is the energy of the state $|n_1, n_2, n_3\rangle$? How many linearly independent states are there with energy $E = (5/2)\hbar\omega$? [That is, what is the degeneracy of the degenerate subspace of states we are studying?]

(b) Express the perturbing Hamiltonian H_1 in terms of creation and annihilation operators.

(c) What is the matrix representation of H_1 in the degenerate subspace you described in part (a)?

(d) What are the eigenvalues and eigenstates of H_1 in the degenerate subspace? What are the eigenvalues and eigenstates of $H = H_0 + H_1$ in the degenerate subspace?

(e) What is the matrix representation of $H_0 + H_1$ in the degenerate subspace if you use the eigenvectors of H_1 as a new basis? (I.e. instead of the original $|n_1, n_2, n_3\rangle$ basis.)

[Note: As we shall see in part (f), this problem is "too simple" in important ways. The aspect of this problem which *will* generalize when we consider more generic perturbations is that if a perturbation breaks a degeneracy, then even an arbitrarily small but nonzero perturbation has qualitative consequences: it selects one particular choice of energy eigenvectors, within thepreviously degenerate subspace. In the present problem, this can be described as follows: if K were initially zero and you were happily using the $|n_1, n_2, n_3\rangle$ states as your basis of energy eigenstates, and then somebody "turns on" a very small but nonzero value of K, this forces you to make a qualitative change in your basis states. The "rotation" you must make from your previous energy eigenstates to the new states which are now the only possible choice of energy eigenstates is not a small one, even though K is arbitrarily small.]

(f) $|\psi\rangle$ and $|\phi\rangle$ are eigenstates of H_0 with *different* energy eigenvalues. That is, $|\psi\rangle$ and $|\phi\rangle$ and belong to different degenerate subspaces. Show that $\langle\phi|H_1|\psi\rangle = 0$ for any two such states. Relate this fact to a statement you can make about the operators H_0 and H_1, without reference to states.

[The fact that $\langle\phi|H_1|\psi\rangle = 0$ if $|\psi\rangle$ and $|\phi\rangle$ and belong to different degenerate subspaces means that H_1 is a "non-generic" perturbation of H_0; a more general perturbation would not have this property. It is only for perturbations with this property that the analysis you have done above — which focusses on one degenerate subspace at a time — is complete. Notice also that in order to analyze $H = H_0 + H_1$, we did not have to assume that K was in any sense small. If H_1 were "generic", we would have had to assume that K was small in order to make progress.]

4. **A Delta-Function Interaction Between Two Bosons in an Infinite Square Well (8 points)**

Do Griffiths problem 6.3.

[The integrals that come up in this problem can certainly all be done by hand; however, there is also nothing wrong with doing them by Mathematica or the equivalent.]

5. **Anharmonic Oscillator (14 points)**

Consider the anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^3 ,$$

treating the λx^3 term as a perturbation. [Hint: you should not find yourself doing any integrals as you do this problem; you should find yourself manipulating harmonic oscillator creation and annihilation operators and harmonic oscillator energy eigenstates.]

(a) Show that the first order shift in the ground state energy is zero. Calculate the shift to order λ^2.

(b) Calculate the ground state wave function to order λ. (You may just write your answer as a sum of harmonic oscillator states.)

(c) Sketch the potential $V(x)$ as a function of x for small λ. Is the state you found in (b) anything like the true ground state? What effect has perturbation theory failed to find?

(d) Now consider instead an anharmonic oscillator with

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} + \lambda x^4 \ ,$$

treating the λx^4 term as a perturbation. Calculate the energy of the ground state to order λ. Sketch $V(x)$ for λ small and positive and for λ small and negative, and comment on what perturbation theory has told you in each case, and in each case comment on whether you think that perturbation theory has given a good approximation to the true ground state energy.