Quantum Physics III (8.06) Spring 2006
Assignment 7

March 27, 2006 **Due Tuesday April 11, 2006**

- Please remember to put your name **and section time** at the top of your paper.
- This problem set is due in a week, as usual. The next problem set, Problem Set 8, will be available before Tuesday April 11 as usual but will be due on Tuesday April 25. You will have 2 weeks for Problem Set 8 because in the interim the first draft of your term paper is due.
- **The Final Exam will be on Monday May 22, 1:30-4:30pm, in Johnson Ice Rink.**
- If you want further problems, beyond those I assign below, with which to teach yourself about the variational method, prepare for the final exam, gain physics intuition and learn some new applications, you could try Griffiths' Problems 7.7, 7.15, 7.16, 7.17, 7.18, 7.19, 7.20.

Readings

The reading assignment for this problem set and the first part of the next one is:

- Griffiths Chapter 7.
- Cohen-Tannoudji Chapter XI, Complements E (scanned copy available at 8.06 website).
- Griffiths Chapter 8 and the Supplementary Notes on the Connection Formulae. (You do not need the latter for this problem set.)

Problem Set 7

1. **Variational bound on the ground state in a power-like potential (10 points)**

   Consider a particle of mass m moving in the one dimensional potential

   $$V(x) = \lambda x^4 \tag{1}$$

   We can obtain an upper bound on the energy of the ground state using the variational method. To find a trial wave function, we regard (1) as a harmonic oscillator potential with a space-dependent frequency, i.e. $V(x) = \frac{1}{2}m\omega^2(x)x^2$

with $\frac{1}{2}m\omega^2(x) = \lambda x^2$. This motivates us to choose the trial wave function to resemble the ground state of a harmonic oscillator potential, i.e.

$$\psi(x) = Ae^{-bx^2}$$

Find the value of b that minimizes $\langle\psi|H|\psi\rangle$ and obtain an upper bound on the ground state energy. (Hint: normalize ψ first to express A in terms of b.)

2. Variational bound on the excited states (15 points)

(a) Prove the following corollary to the variational principle: if $\langle\psi|\psi_{gs}\rangle = 0$, then $\langle\psi|H|\psi\rangle \geq E_{fe}$, where $|\psi_{gs}\rangle$ denotes the ground state wave function and E_{fe} is the energy of the first excited state.

In general it is difficult to be sure a state $|\psi\rangle$ is orthogonal to $|\psi_{gs}\rangle$ since the latter is generally not known exactly. However, if the potential $V(x)$ has some symmetry, it is often possible to realize $\langle\psi|\psi_{gs}\rangle = 0$. For example, if the potential $V(x)$ is an even function of x, then the ground state wave function should also be even in x and thus an odd trial function will be automatically orthogonal to ψ_{gs}.

(b) Give an example of a set of trial wave functions, specified by a single parameter, that could be used to obtain an upper bound on the first excited state energy of (1). Explain your reasoning for choosing the ansatz. But do not go further than writing down the ansatz.

(c) Choose a set of trial wave functions, specified by a single parameter to obtain an upper bound on the first excited state energy of the one-dimensional harmonic oscillator $V(x) = \frac{1}{2}m\omega^2x^2$.

3. Variational bound on the ground state in an exponential potential (15 points)

Unlike in one dimension, an attractive potential in three dimensions does not always have a bound state. A simple variational guess can give us an estimate of how strong a potential must be in order to have a bound state, even though the exact solution would require solving the Schrödinger equation numerically.

Consider a particle of mass m moving in three dimensions under a central force derived from an exponential potential,

$$V(r) = -\alpha e^{-2\mu r} ,$$

where α and μ are positive. Take a simple exponential variational *ansatz* for the ground state wavefunction:

$$\psi_\lambda(r) = Ce^{-\lambda r} . \tag{2}$$

(a) Find the constant C by demanding that $\int d^3r|\psi_\lambda(r)|^2 = 1$.

(b) Compute the variational estimate of the energy of ψ, as a function of λ.

Hint: Once you have normalized the wave function, the variational estimate is given by

$$E(\lambda) = \int d^3r \left\{ \frac{\hbar^2}{2m} \left| \frac{d\psi_\lambda(r)}{dr} \right|^2 + V(r)\,|\psi_\lambda(r)|^2 \right\} \tag{3}$$

Hint: The only integral needed is $\int_0^\infty dx x^n e^{-x} = n!$.

Answer: $E(\lambda) = \frac{\lambda^2}{2m} - \alpha(\frac{\lambda}{\mu+\lambda})^3$.

(c) Show that for small α, the minimum value of $E(\lambda)$ is zero and is obtained for $\lambda = 0$. Interpret this result (for example, where is the particle found when $\lambda = 0$?).

(d) Lets scale out some of the dimensionful parameters to make this problem easier to analyze. Consider $\mathcal{E} = \frac{mE}{\mu^2}$. Show that $\mathcal{E}$ can be written as a function of $x = \lambda/\mu$ and a scaled strength of the potential, $\kappa = \alpha m/\mu^2$. Rewrite the result of part (b) as $\mathcal{E}(\kappa, x)$. Analyze this equation graphically or numerically and find the minimum value of κ for which a bound state exists. What is the value of x at this value of κ.

(e) Does the result of the previous section give you a minimum value of α (for fixed m and μ) required for a bound state, or a maximum, or neither? Explain.

4. **Tunnelling and the Stark Effect (20 points)**

The Stark effect concerns the physics of an atom in an electric field. In this problem, we discuss the possibility that in an electric field, the electron in an atom can tunnel out of the atom, making the atomic bound states unstable. We consider this effect in a simpler one-dimensional analog problem.

Supose an electron is trapped in a one-dimensional square well of depth V_0 and width d:

$$\begin{aligned} V(x) &= -V_0 \text{ for } |x| < d/2 \\ &= 0 \quad \text{for } |x| \geq d/2 \,. \end{aligned}$$

Suppose a weak constant electric field in the x-direction with strength $\mathcal{E}$ is turned on. That is $V \to (V - e\mathcal{E}x)$. Assume throughout this problem that $e\mathcal{E}d \ll \hbar^2/2md^2 \ll V_0$.

(a) Set $\mathcal{E} = 0$ in this part of the problem. Estimate the ground state energy (ie the amount by which the ground state energy is above the bottom of the potential well) by pretending that the well is infinitely deep. (Because $\hbar^2/2md^2 \ll V_0$, this is a good approximation.) Use this estimate of the ground state energy in subsequent parts of the problem.

[Aside: the true ground state energy is lower than what you've estimated. (You can show this, but that's optional.) This means that the tunnelling lifetime you estimate below is an underestimate.]

(b) Sketch the potential with $\mathcal{E} \neq 0$ and explain why the ground state of the $\mathcal{E} = 0$ potential is no longer stable when $\mathcal{E} \neq 0$.

(c) Use the semiclassical approximation to calculate the barrier penetration factor for the ground state. [You should use the fact that $e\mathcal{E}d \ll \hbar^2/2md^2$ to simplify this part of the problem.]

(d) Use classical arguments to convert the barrier penetration factor into an estimate of the lifetime of the bound state.

(e) Now, lets put in numbers. Calculate the lifetime for $V_0 = 20$ eV, $d = 2 \times 10^{-8}$ cm and an electric field of 7×10^4 V/cm. Compare the lifetime you estimate to the age of the universe.

(f) Show that the lifetime goes like $\exp 1/\mathcal{E}$, and explain why this result means that this "instability" could not be obtained in any finite order of perturbation theory, treating $\mathcal{E}$ as a perturbation to the Hamiltonian.