Quantum Physics III (8.06) Spring 2006
Assignment 9

April 22, 2006 **Due FRIDAY May 5, 2006**

- Please remember to put your name **and section time** at the top of your paper.
- Your final paper is due on **Thursday May 4** in lecture.
- Note that Problem Set 10 will be due on FRIDAY May 12.

Readings

- The article by Wick Haxton on the Solar Neutrino Problem is *optional* extra reading.
- The paper by Bitter et al on experimental verification of Berry's phase is *optional* extra reading.
- Read Griffiths Chapter 9 on Time Dependent Perturbation Theory.
- Scattering theory: Griffiths Chapter 11, Ohanian Chapter 11 (posted on the web) and Prof. Jaffe's notes on scattering (posted on the web).

1. **Brick in a Square Well (6 points)**

   Here is a simple enough time dependent perturbation of a simple enough system that everything can be computed analytically.

   Do Griffiths Problem 9.18. (Note: Problem 9.17 in 1st. Ed.)

2. **A Time-Dependent Two-State System (12 points)**

   Consider a two-state system with Hamiltonian

   $$H(t) = \begin{pmatrix} +E & v(t) \\ v(t) & -E \end{pmatrix}$$

   where $v(t)$ is real and where $v \to 0$ for $t \to \pm\infty$.

   (a) Suppose that at $t = -\infty$ the system is in the state $|1\rangle$. Use time dependent perturbation theory to determine the probability that at $t = +\infty$ the system is in the state $|2\rangle$, to lowest order in v.

(b) If $E = 0$, the eigenstates of $H(t)$ do not depend on t. Use this fact to calculate the probability of a transition from $|1\rangle$ to $|2\rangle$ *exactly*, in this case. What is the result obtained from time-dependent perturbation theory in this case? What is the condition that the perturbative result is a good approximation to the exact result?

3. **Excitation of a hydrogen atom (8 points)**

A hydrogen atom is placed in an electric field $\vec{E}(t)$ that is uniform and has the time dependence,

$$\begin{aligned} \vec{E}(t) &= 0 && t < 0 \\ &= \vec{E}_0 e^{-\gamma t} && t > 0 \end{aligned} \tag{1}$$

What is the probability that as $t \to \infty$, the hydrogen atom, initially in the ground state, makes a transition to the $2p$ state?

(*In this problem, you will need matrix elements* $\langle 100|z|21m\rangle$ *for* $m = 0, \pm 1$. *Write each matrix element in a form with a single undetermined numerical factor. You do not need to compute the numerical factors.*)

4. **Decay of the three dimensional harmonic oscillator (12 points)**

The object of this problem is to calculate the lifetime of a charged particle (charge q, mass m) in the first p-state of the three dimensional harmonic oscillator (frequency ω).

a) Write down an expression for the transition rate per unit time, $\Gamma(2p \to 1s)$, for the particle to *spontaneously* emit electromagnetic radiation and make a transition to the ground state. Γ should depend on the frequency of the emitted light and on the matrix element of the operator $q\vec{r}$.

Note that the $2p$ state is three-fold degenerate: it has $\ell = 1$ and can have $m_\ell = -1, 0, 1$.

b) Show that the transition rate *is independent of* m_ℓ.

c) Finally, give a formula for $\Gamma(2p \to 1s)$ in terms of m, ω, q, and fundamental constants.

d) What is the relationship between the transition rate per unit time and the "lifetime" of the $2p$ state?

5. **A wave front crossing a bound particle (12 points)**

Consider a particle in one dimension moving under the influence of some time-independent potential, $V(x)$. Assume that you know the energy levels and

corresponding eigenfunctions for this problem. We now subject the particle to a traveling pulse represented by a space- and time-dependent potential,

$$V(t) = a\delta(x - ct)$$

where $\delta(x)$ is a Dirac δ-function.

(a) Suppose as $t \to -\infty$ the particle is known to be in the ground state whose wavefunction is $\langle x|i\rangle = u_i(x)$. Find the probability for finding the system in some excited state, with wavefunction $\langle x|f\rangle = u_f(x)$ as $t \to \infty$.

(b) Reinterpret your result in part (a) as follows. Regard the δ-function pulse as a superposition of harmonic perturbations, by recalling that the δ function can be represented as a superposition of exponentials:

$$\delta(x - ct) = \frac{1}{2\pi c}\int_{-\infty}^{\infty} d\omega e^{i\omega(x/c-t)}. \tag{2}$$

Show that if you treat each frequency component of the δ function separately, using for each the result we obtained in lecture for a harmonic perturbation (namely that there is a transition if and only if $\omega = \omega_{fi}$ and the amplitude of that transition is the matrix element of the operator coefficient of the harmonic time dependence between the initial and final states) then you get the same result as in part (a).

The lesson is that the analysis we did in lecture with a harmonic time dependence can be applied to very different dime dependences via Fourier transformation.

(c) Apply the result of part (a) to the one dimensional (infinite) square well,

$$\begin{aligned} V(x) &= 0 && \text{for } 0 < x < d, \\ &= \infty && \text{for } x < 0 \text{ or } x > d \end{aligned} \tag{3}$$

Express the probability to transition from the ground state to the first excited state as a function of the dimensionless parameters $\alpha = \frac{a}{\hbar c}$ and $\beta = \frac{d\Delta E}{2\pi\hbar c}$, where $\Delta E = \frac{3\pi^2\hbar^2}{2md^2}$. Show that the transition probability has a maximum for $\beta \approx 1$. Explain this in terms of the time it takes light to cross the potential well and the natural timescale of the quantum system.