Quantum Physics III (8.06) Spring 2006
Solution Set 1

February 14, 2006

For convenience we quote here the conversion formulae between cgs and natural units. From natural units to cgs we convert using

$$\begin{aligned} 1c &= 2.99792458 \times 10^{10}\text{cm/sec}, \\ 1\hbar &= 1.05457266 \times 10^{-27}\text{gm cm}^2/\text{sec}, \\ 1\text{eV} &= 1.602 \times 10^{-12}\text{gm cm}^2/\text{sec}^2. \end{aligned} \tag{1}$$

Conversely, from cgs to natural units we convert using

$$\begin{aligned} 1\text{sec} &= 1.51926689 \times 10^{15}\ \frac{\hbar}{\text{eV}}, \\ 1\text{cm} &= 5.06772886 \times 10^{4}\ \frac{\hbar c}{\text{eV}}, \\ 1\text{gm} &= 5.06958616 \times \frac{\text{eV}}{c^2}. \end{aligned} \tag{2}$$

Also note that:

1 atmosphere $= 1.01 \times 10^6$ dynes/cm^2 (cgs) $= 4.8 \times 10^3\,\text{eV}^4$

1. **Natural units**

   (1 points for each correct answer)

   The conductance ρ has dimensions of electric current divided by voltage. Therefore

   $$[\rho] = [I/V] = \left[\text{charge}^2/(\mathcal{E}\ t)\right] = \frac{\hbar c}{\hbar} = c$$

   where $\mathcal{E}$ denotes energy.

   The units for magnetic flux are those of magnetic field times area, we have

   $$[\phi_B] = [Bl^2] = (\hbar c)^{1/2}.$$

   Since $\mathcal{E} = \vec{\mu}.\vec{B}$,

   $$[\mu] = [\mathcal{E}/B] = \text{e}V^{-1}\,\hbar^{3/2}\,c^{3/2}$$

2. **Magnetic Moments**

   (3 points)

   $$\vec{\mu} \propto e\vec{S}$$

   Introduce m, $\hbar$ and c to fix the dimensions (work in natural units for convenience):

   $$[e\vec{S}] = (\hbar c)^{1/2}\hbar = \hbar^{3/2}c^{1/2}$$

but

$$[\vec{\mu}] = \mathrm{e}V^{-1}\,\hbar^{3/2}\,c^{3/2}.$$

Divide by mc to get the right dimension,

$$\left[\frac{e\vec{S}}{mc}\right] = \mathrm{e}V^{-1}\,\hbar^{3/2}\,c^{3/2}.$$

Therefore,

$$\vec{\mu} = \frac{g}{2}\,\frac{e}{mc}\vec{S}$$

(2 point)

Using conversion formulae (1) we can write the conversion factor

$$1\frac{\mathrm{eV}^2}{(\hbar c)^{3/2}} = \frac{1.602^2 \times 10^{-24}}{(2.99 \times 1.0510^{-17})^{3/2}}\frac{\mathrm{gm}^{1/2}}{\mathrm{cm}^{1/2}\mathrm{sec}} = 14.51\frac{\mathrm{gm}^{1/2}}{\mathrm{cm}^{1/2}\mathrm{sec}} \tag{3}$$

therefore $10^5\,gauss = 6906.08\ \mathrm{e}V^2\,\hbar^{-3/2}\,c^{-3/2}$.

(3 points)

$$\Delta\mathcal{E} = \vec{\mu_\uparrow}.\vec{B} - \vec{\mu_\downarrow}.\vec{B} = 2\frac{g}{2}\frac{eB}{mc}\frac{\hbar}{2}$$

Now set $\hbar = c = 1$ and use $g \approx 2$, $e = 1/\sqrt{137}$ and $m = 511keV$, to get

$$\Delta\mathcal{E} = 1.16 \times 10^{-3}\mathrm{e}V$$

3. **Electron-positron pair production by an electric field**

(a) (3 points) In natural units $m_e c^2 = 511$ keV Therefore we have

$$\begin{aligned} E_0 =& \frac{(m_e c^2)^2}{(\hbar c)^{3/2}} = (511 \times 10^3)^2\frac{\mathrm{eV}^2}{(\hbar c)^{3/2}} \\ =&(511 \times 10^3)^2\frac{e}{(\hbar c)^{1/2}}\,\frac{\mathrm{Volts}}{\mathrm{cm}}\,\frac{\mathrm{eVcm}}{\hbar c} \\ =&(511 \times 10^3)^2\,\sqrt{\alpha}\,(5.07 \times 10^4)\frac{\mathrm{Volts}}{\mathrm{cm}} \\ =&1.13 \times 10^{15}\frac{\mathrm{Volts}}{\mathrm{cm}} \end{aligned}$$

where α is fine structure constant and we have used the second equation in (2).

(b) (5 points) Imagine an electron traveling the distance $\delta x = c\delta t = \frac{\hbar c}{\delta\epsilon} = \frac{\hbar c}{2m_e c^2}$. The work done by the electric field $\vec{E}_c$ is

$$\delta W = eE_c\delta x = \frac{eE_c\hbar c}{2m_e c^2}.$$

To compensate for the rest mass of the two electrons this work should equal to $2m_e c^2$. Therefore

$$E_c = \frac{4m_e^2c^4}{e\hbar c} = 4\left(\frac{\hbar c}{e^2}\right)^{1/2}\frac{(m_e c^2)^2}{(\hbar c)^{3/2}} = \frac{4}{\alpha^{1/2}}E_0 \approx 46.8\,E_0$$

where we recalled that $\alpha = e^2/\hbar c = 1/137$. Thus the ratio of the critical vacuum breakdown electric field to the electric field associated with the electron's rest energy is

$$\frac{E_c}{E_0} = \frac{4}{\sqrt{\alpha}} \approx 46.8.$$

Due to that the electromagnetic interaction is weak (i.e. $\alpha \ll 1$), it takes a larger electric field (with an additional factor $1/\sqrt{\alpha}$) to produce electron and positron pair than the naive guess $4E_0$.

4. **The Bag Pressure**

(a) (2 point) Since $[pR] = \hbar$ and $[E] = [p]c$, eliminating $[p]$ we get $[E] = \hbar c/[R]$. Hence:

$$E(R) = \frac{3\hbar c}{R} + B\frac{4\pi R^3}{3}$$

(b) (5 points) Since $R = R_0$ minimizes $E(R)$, therefore

$$\left.\frac{dE(R)}{dR}\right|_{R=R_0} = -\frac{3}{R_0^2} + 4\pi R_0^2 B = 0$$

$$\Rightarrow R_0 = \left(\frac{3}{4\pi B}\right)^{1/4}$$

Substitute R_0 back into $E(R)$,to get M:

$$M = E(R_0) = 4\left(\frac{4\pi}{3}B\right)^{1/4}$$

(c) (4 points) Invert result of previous part, to get

$$B = \frac{3}{4\pi}\left(\frac{M}{4}\right)^4$$

Substitute for $M = 940\,\mathrm{Me}V$ to obtain $B = 7.28 \times 10^8\,\mathrm{Me}V^4$. Use $197\,\mathrm{Me}V\,\mathrm{fm} = 1$ to get:

$$B = \frac{7.28 \times 10^8}{(197)^3}\frac{\mathrm{Me}V}{\mathrm{fm}^3} = 95\,\mathrm{Me}V/\mathrm{fm}^3$$

(d) (1 point) The units of energy per unit volume and that of pressure are both

$$\frac{gm}{cm\,\mathrm{s}^2} = \frac{\mathrm{eV}^4}{(\hbar c)^3}$$

(e) (3 points) The bag constant B has units of energy density which, as we just demonstrated, is the same as pressure. We need to convert from natural units into CGS.

$$\begin{aligned} B = \frac{95\,\mathrm{Me}V}{\mathrm{fm}^3} = 95\frac{1.602 \times 10^{-6}\,\mathrm{ergs}}{(10^{-13}\,cm)^3} &= 1.5 \times 10^{35}\,\mathrm{dynes}/cm^2 \\ &= 1.5 \times 10^{29}\,\mathrm{atm}. \end{aligned}$$

This pressure is extraordinarily large—the strong force is very strong.

(f) (3 points) From result of part (c) $B = 0.48\,\mathrm{fm}^{-4}$. Using expression of R_0 obtained in part (b), we get $R_0 = 0.84\,\mathrm{fm}$. This value is just slightly smaller then $1\,\mathrm{fm}$, which is typically the range of strong interactions.

5. **The Accelerating Universe**

(a) (3 points) The cosmological constant has the unit of an energy density. The natural scale for the cosmological constant Λ is thus $M_{\mathrm{Plank}}/L^3_{\mathrm{Plank}}$. M_{Plank} and L_{Plank} are the natural scales of mass and length in the unit system of $\hbar$, c and G_N, which are

$$M_{\mathrm{Plank}} = \sqrt{\frac{\hbar c}{G_N}}$$
$$L_{\mathrm{Plank}} = \sqrt{\frac{\hbar G_N}{c^3}}$$

We can now find the natural value of Λ:

$$M_{\mathrm{Plank}}/L^3_{\mathrm{Plank}} = 2.9 \times 10^{126}\mathrm{eV/cm}^3$$

(b) (2 point) In Planck units, the observed value is $\Lambda = 6.9 \times 10^{-124}$, which is an extremely tiny number.

6. **Fermi energy, velocity and temperature of copper**

(a) (2 points) The Fermy energy of copper is

$$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$$

where ρ is the number of free electrons per unit volume. For copper there is one free electron per atom, therefore

$$\rho = \frac{N_A d}{A} = \frac{8.96\,gm/cm^3}{63.5\,gm/\mathrm{mole}}\,\frac{6.02\times 10^{23}\,\mathrm{atoms}}{\mathrm{mole}} = 8.49\times 10^{22}\,\mathrm{atoms}/cm^3$$

where N_A is the Avogadro's number, d is the density of copper and A is the atomic weight of copper. Substituting numbers we obtain

$$E_F \approx 7.1eV \approx 1.12\times 10^{-11}\mathrm{erg}.$$

(b) (2 points)

The corresponding Fermi velocity can be found from the relationship $E_F = \frac{1}{2}mv_F^2$. We have

$$v_F = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\times 7.1}{511\times 10^3}}\,c = 5.3\times 10^{-3}\,c = 1.57\times 10^8\mathrm{cm/sec}$$

Since the Fermi velocity is much smaller than the speed of light we can safely assume that electrons in the copper crystal are nonrelativistic. Note that the obtained Fermi v_F velocity is about $c/137$, i.e. is of order of $e^2/\hbar$, the electron velocity in hydrogen atom. This agrees with what one expects on dimensional grounds.

(c) (2 points) The Fermi temperature is given by

$$T_F = \frac{E_F}{k_B} \approx 81 \times 10^3 \mathrm{K} = 7.1 eV.$$

Therefore we can approximate the electron gas to be at zero temperature.

(d) (2 points) The degeneracy pressure is

$$P = \frac{(3\pi^2)^{\frac{2}{3}}\hbar^2}{5m}\rho^{\frac{5}{3}} \approx 3.80 \times 10^{11} \frac{\mathrm{gm}}{\mathrm{cm\ sec}^2} \approx 3.8 \times 10^5 \mathrm{atm}.$$